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ANNOUNCEMENT OF THE PUBLICATION OF NEW AND MORE EXTENSIVE TABLES OF LAGRANGEAN INTERPOLATION COEFFICIENTS

Any fourth degree algebraic polynomial can be expressed in the form

$$f(x) = A_{-2}f(a_{-2}) + A_{-1}f(a_{-1}) + A_0 f(a_0) + A_1 f(a_1) + A_2 f(a_2)$$

where the points a_{-2}, a_{-1}, a_0, a_1, a_2 are uniformly spaced at intervals h, where the Lagrangean interpolation coefficients A_{-2}, A_{-1}, A_0, A_1, A_2 are fourth degree polynomials in p, and where p is defined by the relation $x = a_0 + ph$. p is thus the distance, measured with h as a unit, at which x lies to the right of the midpoint a_0. Tables of these five point coefficients computed to seven decimal places for consecutive values of p which differ by .001 and which cover the range $-2 \leqslant p \leqslant 2$, corresponding to $a_{-2} < x < a_2$, have been computed and are now available. These tables were computed at the request of the Chief of Ordnance, U. S. Army (0.0. 063.2/33, July 24, 1940, from the Chief of Ordnance to Dr. Arnold N. Lowan, Project Supervisor for Works Projects Administration for New York City) by the project for "Computation of Mathematical Tables," O.P. 65–2–97–33, conducted by the Works Projects Administration for New York City under the sponsorship of the National Bureau of Standards, Dr. Lyman J. Briggs, Director.

These 5-point tables were originally issued in June, 1941 for private distribution in the War Department. They have just been released for public reference, but inasmuch as the volume of tables which is to include these 5-point values will not be ready for some time, permission has been obtained by the Marchant Calculating Maching Company to reproduce them in an advance form.

By writing to the Marchant Calculating Machine Company, Oakland, California, for publication number MM-228, copies of these Tables may be obtained free of charge by members of any of the recognized Mathematical, Scientific, or Engineering Societies, by members of the faculties of Educational Institutions, and by Public Libraries. They are available to others at a price of fifteen cents.

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TABLE OF THE INTEGRALS $\int_0^x J_0(t)dt$ and $\int_0^x Y_0(t)dt$

By Arnold N. Lowan and Milton Abramowitz*

In the course of the computation of the functions $J_0(z)$, $J_1(z)$ and $Y_0(z)$, $Y_1(z)$ for complex arguments, the values of the derivatives of $J_0(z)$ were obtained for real arguments from 0 to 10 at intervals of 0.01. This made it possible to compute the values of the first of the above two integrals by a combination of Taylor and MacLaurin expansions.

For the second integral, we have

$$\int_0^x Y_0(t)dt = \frac{2}{\pi}\left\{\gamma + \log\frac{x}{2}\right\}\int_0^x J_0(t)dt - S_1 - S_2$$

$$S_1 = \frac{2}{\pi}\sum_{r=0}^{\infty}(-1)^r\frac{(\frac{1}{2}x)^{2r}}{(2r+1)(r!)^2} = \int_0^x\left\{\frac{1}{t}\int_0^t J_0(u)du\right\}dt$$

$$S_2 = \frac{2}{\pi}\sum_{r=1}^{\infty}(-1)^r\frac{(\frac{1}{2}x)^{2r+1}}{(2r+1)(r!)^2}\left\{1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots + \frac{1}{r}\right\}$$

The series S_2 was obtained in the same manner as $\int_0^x J_0(t)dt$ as a by-product in the computation of the table $Y_0(z)$ and $Y_1(z)$ for complex arguments. The series S_1 was evaluated by iterated Taylor expansions by a method similar to that described in the introduction to the table of Sine, Cosine and Exponential Integrals, prepared by the Mathematical Tables Project. The following table lists the values of the two integrals for the range between 0 and 10 at intervals of 0.01.

* Members of the Mathematical Tables Project, National Bureau of Standards.

TABLE OF THE INTEGRAL $\int_0^x J_0(t)dt$

x	$\int_0^x J_0(t)dt$	x	$\int_0^x J_0(t)dt$	x	$\int_0^x J_0(t)dt$	x	$\int_0^x J_0(t)dt$
0.00	0.00000 00000	0.50	0.48968 05066	1.00	0.91973 04101	1.50	1.24144 95144
01	.00999 99167	51	.49905 30157	01	.92736 03304	51	.24653 98711
02	.01999 93333	52	.50840 08451	02	.93494 59223	52	.25157 42967
03	.02999 77501	53	.51772 35435	03	.94248 68658	53	.25655 26573
04	.03999 46670	54	.52702 06618	04	.94998 28437	54	.26147 48232
0.05	0.04998 95843	0.55	0.53629 17525	1.05	0.95743 35426	1.55	1.26634 06686
06	.05998 20024	56	.54553 63702	06	.96483 86520	56	.27115 00719
07	.06997 14219	57	.55475 40714	07	.97219 78651	57	.27590 29155
08	.07995 73436	58	.56394 44147	08	.97951 08783	58	.28059 90859
09	.08993 92684	59	.57310 69606	09	.98677 73916	59	.28523 84736
0.10	0.09991 66979	0.60	0.58224 12719	1.10	0.99399 71082	1.60	1.28982 09734
11	.10988 91336	61	.59134 69132	11	1.00116 97348	61	.29434 64839
12	.11985 60777	62	.60042 34515	12	.00829 49817	62	.29881 49080
13	.12981 70327	63	.60947 04558	13	.01537 25625	63	.30322 61526
14	.13977 15013	64	.61848 74976	14	.02240 21945	64	.30758 01287
0.15	0.14971 89872	0.65	0.62747 41501	1.15	1.02938 35984	1.65	1.31187 67515
16	.15965 89942	66	.63642 99894	16	.03631 64985	66	.31611 59402
17	.16959 10268	67	.64535 45933	17	.04320 06226	67	.32029 76182
18	.17951 45901	68	.65424 75424	18	.05003 57021	68	.32442 17129
19	.18942 91899	69	.66310 84194	19	.05682 14721	69	.32848 81559
0.20	0.19933 43325	0.70	0.67193 68094	1.20	1.06355 76711	1.70	1.33249 68829
21	.20922 95252	71	.68073 23001	21	.07024 40416	71	.33644 78337
22	.21911 42756	72	.68949 44814	22	.07688 03295	72	.34034 09523
23	.22898 80926	73	.69822 29459	23	.08346 62843	73	.34417 61867
24	.23885 04855	74	.70691 72885	24	.09000 16593	74	.34795 34890
0.25	0.24870 09646	0.75	0.71557 71069	1.25	1.09648 62116	1.75	1.35167 28157
26	.25853 90413	76	.72420 20012	26	.10291 97019	76	.35533 41271
27	.26836 42276	77	.73279 15740	27	.10930 18946	77	.35893 73878
28	.27817 60365	78	.74134 54308	28	.11563 25580	78	.36248 25664
29	.28797 39824	79	.74986 31795	29	.12191 14640	79	.36596 96358
0.30	0.29775 75802	0.80	0.75834 44308	1.30	1.12813 83885	1.80	1.36939 85727
31	.30752 63462	81	.76678 87981	31	.13431 31110	81	.37276 93584
32	.31727 97978	82	.77519 58976	32	.14043 54148	82	.37608 19778
33	.32701 74534	83	.78356 53482	33	.14650 50873	83	.37933 64202
34	.33673 88327	84	.79189 67715	34	.15252 19193	84	.38253 26790
0.35	0.34644 34566	0.85	0.80018 97922	1.35	1.15848 57059	1.85	1.38567 07517
36	.35613 08472	86	.80844 40376	36	.16439 62457	86	.38875 06397
37	.36580 05279	87	.81665 91379	37	.17025 33415	87	.39177 23489
38	.37545 20235	88	.82483 47264	38	.17605 67998	88	.39473 58889
39	.38508 48601	89	.83297 04390	39	.18180 64310	89	.39764 12735
0.40	0.39469 85653	0.90	0.84106 59149	1.40	1.18750 20495	1.90	1.40048 85208
41	.40429 26679	91	.84912 07962	41	.19314 34735	91	.40327 76527
42	.41386 66984	92	.85713 47277	42	.19873 05253	92	.40600 86954
43	.42342 01886	93	.86510 73577	43	.20426 30311	93	.40868 16789
44	.43295 26722	94	.87303 83372	44	.20974 08209	94	.41129 66375
0.45	0.44246 36839	0.95	0.88092 73205	1.45	1.21516 37289	1.95	1.41385 36095
46	.45195 27606	96	.88877 39649	46	.22053 15932	96	.41635 26373
47	.46141 94404	97	.89657 79308	47	.22584 42558	97	.41879 37671
48	.47086 32633	98	.90433 88819	48	.23110 15628	98	.42117 70494
49	.48028 37709	99	.91205 64850	49	.23630 33643	99	.42350 25386
0.50	0.48968 05066	1.00	0.91973 04101	1.50	1.24144 95144	2.00	1.42577 02932

TABLE OF THE INTEGRAL $\int_0^x J_0(t)dt$—*Continued*

x	$\int_0^x J_0(t)dt$	x	$\int_0^x J_0(t)dt$	x	$\int_0^x J_0(t)dt$	x	$\int_0^x J_0(t)dt$
2.00	1.42577 02932	2.50	1.46798 09446	3.00	1.38756 72520	3.50	1.22330 57382
01	.42798 03757	51	.46747 22934	01	.38494 98418	51	.21949 76619
02	.43013 28524	52	.46691 41820	02	.38229 88997	52	.21567 62671
03	.43222 77940	53	.46630 68621	03	.37961 48016	53	.21184 19735
04	.43426 52748	54	.46565 05890	04	.37689 79247	54	.20799 52004
2.05	1.43624 53733	2.55	1.46494 56209	3.05	1.37414 86481	3.55	1.20413 63671
06	.43816 81718	56	.46419 22191	06	.37136 73527	56	.20026 58931
07	.44003 37568	57	.46339 06480	07	.36855 44205	57	.19638 41975
08	.44184 22184	58	.46254 11750	08	.36571 02356	58	.19249 16993
09	.44359 36509	59	.46164 40705	09	.36283 51834	59	.18858 88175
2.10	1.44528 81525	2.60	1.46069 96081	3.10	1.35992 96508	3.60	1.18467 59706
11	.44692 58252	61	.45970 80641	11	.35699 40262	61	.18075 35772
12	.44850 67750	62	.45866 97178	12	.35402 86994	62	.17682 20554
13	.45003 11116	63	.45758 48516	13	.35103 40617	63	.17288 18229
14	.45149 89489	64	.45645 37506	14	.34801 05058	64	.16893 32973
2.15	1.45291 04044	2.65	1.45527 67029	3.15	1.34495 84256	3.65	1.16497 68956
16	.45426 55995	66	.45405 39992	16	.34187 82164	66	.16101 30346
17	.45556 46595	67	.45278 59333	17	.33877 02747	67	.15704 21304
18	.45680 77135	68	.45147 28017	18	.33563 49983	68	.15306 45988
19	.45799 48944	69	.45011 49035	19	.33247 27863	69	.14908 08550
2.20	1.45912 63387	2.70	1.44871 25408	3.20	1.32928 40386	3.70	1.14509 13136
21	.46020 21871	71	.44726 60181	21	.32606 91568	71	.14109 63888
22	.46122 25838	72	.44577 56429	22	.32282 85431	72	.13709 64940
23	.46218 76766	73	.44424 17251	23	.31956 26011	73	.13309 20421
24	.46309 76174	74	.44266 45773	24	.31627 17352	74	.12908 34450
2.25	1.46395 25615	2.75	1.44104 45147	3.25	1.31295 63510	3.75	1.12507 11144
26	.46475 26681	76	.43938 18550	26	.30961 68548	76	.12105 54607
27	.46549 80999	77	.43767 69187	27	.30625 36542	77	.11703 68939
28	.46618 90236	78	.43593 00285	28	.30286 71573	78	.11301 58230
29	.46682 56091	79	.43414 15096	29	.29945 77735	79	.10899 26562
2.30	1.46740 80303	2.80	1.43231 16899	3.30	1.29602 59125	3.80	1.10496 78009
31	.46793 64645	81	.43044 08994	31	.29257 19853	81	.10094 16633
32	.46841 10928	82	.42852 94708	32	.28909 64033	82	.09691 46491
33	.46883 20996	83	.42657 77388	33	.28559 95789	83	.09288 71625
34	.46919 96732	84	.42458 60409	34	.28208 19249	84	.08885 96072
2.35	1.46951 40051	2.85	1.42255 47165	3.35	1.27854 38549	3.85	1.08483 23854
36	.46977 52907	86	.42048 41074	36	.27498 57833	86	.08080 58986
37	.46998 37285	87	.41837 45577	37	.27140 81247	87	.07678 05469
38	.47013 95208	88	.41622 64137	38	.26781 12945	88	.07275 67295
39	.47024 28733	89	.41404 00238	39	.26419 57087	89	.06873 48442
2.40	1.47029 39949	2.90	1.41181 57386	3.40	1.26056 17835	3.90	1.06471 52877
41	.47029 30983	91	.40955 39108	41	.25690 99358	91	.06069 84555
42	.47024 03994	92	.40725 48952	42	.25324 05827	92	.05668 47418
43	.47013 61176	93	.40491 90488	43	.24955 41418	93	.05267 45395
44	.46998 04754	94	.40254 67303	44	.24585 10311	94	.04866 82401
2.45	1.46977 36989	2.95	1.40013 83007	3.45	1.24213 16688	3.95	1.04466 62340
46	.46951 60175	96	.39769 41228	46	.23839 64734	96	.04066 89099
47	.46920 76638	97	.39521 45613	47	.23464 58636	97	.03667 66551
48	.46884 88738	98	.39269 99831	48	.23088 02584	98	.03268 98558
49	.46843 98866	99	.39015 07565	49	.22710 00769	99	.02870 88962
2.50	1.46798 09446	3.00	1.38756 72520	3.50	1.22330 57382	4.00	1.02473 41595

TABLE OF THE INTEGRAL $\int_0^x J_0(t)dt$—*Continued*

x	$\int_0^x J_0(t)dt$	x	$\int_0^x J_0(t)dt$	x	$\int_0^x J_0(t)dt$	x	$\int_0^x J_0(t)dt$
4.00	1.02473 41595	4.50	0.84186 25481	5.00	0.71531 19178	5.50	0.66891 44989
01	.02076 60269	51	.83866 87208	01	.71355 23476	51	.66886 31228
02	.01680 48784	52	.83549 82671	02	.71182 56454	52	.66884 58334
03	.01285 10922	53	.83235 14520	03	.71013 19182	53	.66886 25707
04	.00890 50449	54	.82922 85376	04	.70847 12698	54	.66891 32714
4.05	1.00496 71114	4.55	0.82612 97832	5.05	0.70684 38003	5.55	0.66899 78694
06	1.00103 76651	56	.82305 54453	06	.70524 96067	56	.66911 62949
07	0.99711 70775	57	.82000 57774	07	.70368 87824	57	.66926 84753
08	.99320 57184	58	.81698 10302	08	.70216 14176	58	.66945 43348
09	.98930 39559	59	.81398 14514	09	.70066 75989	59	.66967 37946
4.10	0.98541 21560	4.60	0.81100 72858	5.10	0.69920 74098	5.60	0.66992 67724
11	.98153 06834	61	.80805 87752	11	.69778 09301	61	.67021 31834
12	.97765 99005	62	.80513 61585	12	.69638 82363	62	.67053 29391
13	.97380 01680	63	.80223 96717	13	.69502 94017	63	.67088 59484
14	.96995 18445	64	.79936 95475	14	.69370 44960	64	.67127 21168
4.15	0.96611 52870	4.65	0.79652 60159	5.15	0.69241 35855	5.65	0.67169 13471
16	.96229 08503	66	.79370 93038	16	.69115 67331	66	.67214 35388
17	.95847 88872	67	.79091 96350	17	.68993 39985	67	.67262 85886
18	.95467 97485	68	.78815 72302	18	.68874 54378	68	.67314 63899
19	.95089 37831	69	.78542 23070	19	.68759 11037	69	.67369 68334
4.20	0.94712 13375	4.70	0.78271 50802	5.20	0.68647 10457	5.70	0.67427 98068
21	.94336 27565	71	.78003 57612	21	.68538 53097	71	.67489 51947
22	.93961 83825	72	.77738 45584	22	.68433 39383	72	.67554 28789
23	.93588 85558	73	.77476 16771	23	.68331 69707	73	.67622 27381
24	.93217 36145	74	.77216 73194	24	.68233 44427	74	.67693 46484
4.25	0.92847 38947	4.75	0.76960 16843	5.25	0.68138 63868	5.75	0.67767 84828
26	.92478 97300	76	.76706 49676	26	.68047 28320	76	.67845 41113
27	.92112 14519	77	.76455 73620	27	.67959 38041	77	.67926 14014
28	.91746 93896	78	.76207 90569	28	.67874 93252	78	.68010 02174
29	.91383 38699	79	.75963 02386	29	.67793 94143	79	.68097 04211
4.30	0.91021 52175	4.80	0.75721 10902	5.30	0.67716 40870	5.80	0.68187 18713
31	.90661 37546	81	.75482 17915	31	.67642 33556	81	.68280 44240
32	.90302 98009	82	.75246 25191	32	.67571 72288	82	.68376 79326
33	.89946 36741	83	.75013 34465	33	.67504 57121	83	.68476 22475
34	.89591 56889	84	.74783 47437	34	.67440 88078	84	.68578 72167
4.35	0.89238 61582	4.85	0.74556 65776	5.35	0.67380 65146	5.85	0.68684 26851
36	.88887 53919	86	.74332 91120	36	.67323 88279	86	.68792 84953
37	.88538 36078	87	.74112 25070	37	.67270 57401	87	.68904 44868
38	.88191 13809	88	.73894 69198	38	.67220 72398	88	.69019 04969
39	.87845 87438	89	.73680 25042	39	.67174 33126	89	.69136 63600
4.40	0.87502 60866	4.90	0.73468 94106	5.40	0.67131 39407	5.90	0.69257 19078
41	.87161 37067	91	.73260 77861	41	.67091 91030	91	.69380 69697
42	.86822 18990	92	.73055 77748	42	.67055 87751	92	.69507 13723
43	.86485 09558	93	.72853 95170	43	.67023 29293	93	.69636 49397
44	.86150 11667	94	.72655 31500	44	.66994 15348	94	.69768 74934
4.45	0.85817 28186	4.95	0.72459 88076	5.45	0.66968 45572	5.95	0.69903 88526
46	.85486 61958	96	.72267 66205	46	.66946 19592	96	.70041 88337
47	.85158 15798	97	.72078 67158	47	.66927 36999	97	.70182 72510
48	.84831 92496	98	.71892 92174	48	.66911 97354	98	.70326 39159
49	.84507 94812	99	.71710 42456	49	.66900 00186	99	.70472 86379
4.50	0.84186 25481	5.00	0.71531 19178	5.50	0.66891 44989	6.00	0.70622 12236

TABLE OF THE INTEGRAL $\int_0^x J_0(t)dt$—*Continued*

x	$\int_0^x J_0(t)dt$	x	$\int_0^x J_0(t)dt$	x	$\int_0^x J_0(t)dt$	x	$\int_0^x J_0(t)dt$
6.00	0.70622 12236	6.50	0.81147 67291	7.00	0.95464 03155	7.50	1.09917 14142
01	.70774 14774	51	.81408 53199	01	.95764 12922	51	.10182 80071
02	.70928 92016	52	.81670 90104	02	.96064 24367	52	.10447 08277
03	.71086 41957	53	.81934 75154	03	.96364 34488	53	.10709 96304
04	.71246 62572	54	.82200 05485	04	.96664 40289	54	.10971 41711
6.05	0.71409 51812	6.55	0.82466 78223	7.05	0.96964 38777	7.55	1.11231 42073
06	.71575 07605	56	.82734 90487	06	.97264 26964	56	.11489 94985
07	.71743 27858	57	.83004 39384	07	.97564 01869	57	.11746 98058
08	.71914 10454	58	.83275 22013	08	.97863 60515	58	.12002 48922
09	.72087 53254	59	.83547 35466	09	.98162 99931	59	.12256 45223
6.10	0.72263 54100	6.60	0.83820 76824	7.10	0.98462 17153	7.60	1.12508 84628
11	.72442 10808	61	.84095 43161	11	.98761 09222	61	.12759 64821
12	.72623 21177	62	.84371 31545	12	.99059 73187	62	.13008 83504
13	.72806 82981	63	.84648 39033	13	.99358 06103	63	.13256 38400
14	.72992 93976	64	.84926 62678	14	.99656 05031	64	.13502 27249
6.15	0.73181 51898	6.65	0.85205 99524	7.15	0.99953 67043	7.65	1.13746 47812
16	.73372 54459	66	.85486 46610	16	1.00250 89215	66	.13988 97867
17	.73565 99355	67	.85768 00967	17	.00547 68633	67	.14229 75214
18	.73761 84259	68	.86050 59621	18	.00844 02391	68	.14468 77673
19	.73960 06826	69	.86334 19593	19	.01139 87591	69	.14706 03081
6.20	0.74160 64692	6.70	0.86618 77897	7.20	1.01435 21344	7.70	1.14941 49299
21	.74363 55473	71	.86904 31543	21	.01730 00771	71	.15175 14206
22	.74568 76766	72	.87190 77535	22	.02024 23000	72	.15406 95702
23	.74776 26151	73	.87478 12874	23	.02317 85172	73	.15636 91708
24	.74986 01188	74	.87766 34555	24	.02610 84434	74	.15865 00166
6.25	0.75197 99420	6.75	0.88055 39571	7.25	1.02903 17946	7.75	1.16091 19039
26	.75412 18371	76	.88345 24909	26	.03194 82877	76	.16315 46311
27	.75628 55548	77	.88635 87554	27	.03485 76407	77	.16537 79989
28	.75847 08443	78	.88927 24488	28	.03775 95728	78	.16758 18098
29	.76067 74527	79	.89219 32690	29	.04065 38040	79	.16976 58688
6.30	0.76290 51256	6.80	0.89512 09137	7.30	1.04354 00558	7.80	1.17192 99830
31	.76515 36072	81	.89805 50802	31	.04641 80507	81	.17407 39618
32	.76742 26396	82	.90099 54658	32	.04928 75123	82	.17619 76165
33	.76971 19637	83	.90394 17675	33	.05214 81655	83	.17830 07610
34	.77202 13187	84	.90689 36823	34	.05499 97366	84	.18038 32114
6.35	0.77435 04422	6.85	0.90985 09070	7.35	1.05784 19529	7.85	1.18244 47858
36	.77669 90704	86	.91281 31382	36	.06067 45431	86	.18448 53049
37	.77906 69379	87	.91578 00728	37	.06349 72373	87	.18650 45914
38	.78145 37779	88	.91875 14075	38	.06630 97667	88	.18850 24707
39	.78385 93222	89	.92172 68388	39	.06911 18642	89	.19047 87702
6.40	0.78628 33012	6.90	0.92470 60635	7.40	1.07190 32638	7.90	1.19243 33198
41	.78872 54438	91	.92768 87784	41	.07468 37009	91	.19436 59516
42	.79118 54776	92	.93067 46805	42	.07745 29126	92	.19627 65002
43	.79366 31291	93	.93366 34666	43	.08021 06373	93	.19816 48026
44	.79615 81232	94	.93665 48341	44	.08295 66148	94	.20003 06982
6.45	0.79867 01837	6.95	0.93964 84803	7.45	1.08569 05865	7.95	1.20187 40286
46	.80119 90331	96	.94264 41027	46	.08841 22953	96	.20369 46380
47	.80374 43928	97	.94564 13992	47	.09112 14856	97	.20549 23731
48	.80630 59829	98	.94864 00679	48	.09381 79035	98	.20726 70828
49	.80888 35224	99	.95163 98070	49	.09650 12967	99	.20901 86188
6.50	0.81147 67291	7.00	0.95464 03155	7.50	1.09917 14142	8.00	1.21074 68348

TABLE OF THE INTEGRAL $\int_0^x J_0(t)dt$—*Concluded*

x	$\int_0^x J_0(t)dt$	x	$\int_0^x J_0(t)dt$	x	$\int_0^x J_0(t)dt$	x	$\int_0^x J_0(t)dt$
8.00	1.21074 68348	8.50	1.26494 80240	9.00	1.25226 64460	9.50	1.17944 18392
01	.21245 15875	51	.26535 37589	01	.25135 08640	51	.17749 45237
02	.21413 27356	52	.26573 21734	02	.25041 08697	52	.17553. 12935
03	.21579 01407	53	.26608 32616	03	.24944 65842	53	.17355 23615
04	.21742 36666	54	.26640 70207	04	.24845 81308	54	.17155 79419
8.05	1.21903 31798	8.55	1.26670 34502	9.05	1.24744 56348	9.55	1.16954 82503
06	.22061 85493	56	.26697 25524	06	.24640 92242	56	.16752 35036
07	.22217 96466	57	.26721 43324	07	.24534 90289	57	.16548 39199
08	.22371 63457	58	.26742 87979	08	.24426 51811	58	.16342 97186
09	.22522 85234	59	.26761 59593	09	.24315 78151	59	.16136 11202
8.10	1.22671 60587	8.60	1.26777 58297	9.10	1.24202 70675	9.60	1.15927 83464
11	.22817 88335	61	.26790 84249	11	.24087 30771	61	.15718 16203
12	.22961 67320	62	.26801 37632	12	.23969 59845	62	.15507 11659
13	.23102 96413	63	.26809 18657	13	.23849 59329	63	.15294 72084
14	.23241 74508	64	.26814 27563	14	.23727 30674	64	.15080 99740
8.15	1.23378 00528	8.65	1.26816 64613	9.15	1.23602 75350	9.65	1.14865 96903
16	.23511 73418	66	.26816 30097	16	.23475 94850	66	.14649 65856
17	.23642 92154	67	.26813 24334	17	.23346 90688	67	.14432 08894
18	.23771 55735	68	.26807 47666	18	.23215 64398	68	.14213 28321
19	.23897 63188	69	.26799 00462	19	.23082 17532	69	.13993 26453
8.20	1.24021 13565	8.70	1.26787 83120	9.20	1.22946 51666	9.70	1.13772 05614
21	.24142 05946	71	.26773 96061	21	.22808 68393	71	.13549 68136
22	.24260 39436	72	.26757 39733	22	.22668 69326	72	.13326 16363
23	.24376 13168	73	.26738 14611	23	.22526 56099	73	.13101 52647
24	.24489 26300	74	.26716. 21195	24	.22382 30365	74	.12875 79348
8.25	1.24599 78019	8.75	1.26691 60012	9.25	1.22235 93795	9.75	1.12648 98834
26	.24707 67536	76	.26664 31613	26	.22087 48079	76	.12421 13481
27	.24812 94092	77	.26634 36575	27	.21936 94928	77	.12192 25676
28	.24915 56951	78	.26601 75503	28	.21784 36070	78	.11962 37811
29	.25015 55406	79	.26566 49025	29	.21629 73251	79	.11731 52284
8.30	1.25112 88778	8.80	1.26528 57796	9.30	1.21473 08237	9.80	1.11499 71504
31	.25207 56413	81	.26488 02496	31	.21314 42811	81	.11266 97884
32	.25299 57685	82	.26444 83828	32	.21153 78773	82	.11033 33846
33	.25388 91993	83	.26399 02525	33	.20991 17942	83	.10798 81816
34	.25475 58767	84	.26350 59340	34	.20826 62154	84	.10563 44228
8.35	1.25559 57459	8.85	1.26299 55055	9.35	1.20660 13263	9.85	1.10327 23522
36	.25640 87553	86	.26245 90474	36	.20491 73139	86	.10090 22143
37	.25719 48556	87	.26189 65427	37	.20321 43670	87	.09852 42541
38	.25795 40006	88	.26130 83769	38	.20149 26760	88	.09613 87172
39	.25868 61464	89	.26069 43380	39	.19975 24329	89	.09374 58498
8.40	1.25939 12520	8.90	1.26005 46162	9.40	1.19799 38314	9.90	1.09134 58985
41	.26006 92793	91	.25938 93044	41	.19621 70668	91	.08893 91101
42	.26072 01926	92	.25869 84977	42	.19442 23360	92	.08652 57323
43	.26134 39591	93	.25798 22939	43	.19260 98374	93	.08410 60129
44	.26194 05486	94	.25724 07929	44	.19077 97711	94	.08168 02000
8.45	1.26250 99337	8.95	1.25647 40971	9.45	1.18893 23386	9.95	1.07924 85423
46	.26305 20898	96	.25568 23113	46	.18706 77428	96	.07681 12888
47	.26356 69948	97	.25486 55427	47	.18518 61883	97	.07436 86885
48	.26405 46294	98	.25402 39006	48	.18328 78810	98	.07192 09911
49	.26451 49771	99	.25315 74970	49	.18137 30284	99	.06946 84463
8.50	1.26494 80240	9.00	1.25226 64460	9.50	1.17944 18392	10.00	1.06701 13040

TABLE OF THE INTEGRAL $\int_0^x Y_0(t)dt$

x	$\int_0^x Y_0(t)dt$	x	$\int_0^x Y_0(t)dt$	x	$\int_0^x Y_0(t)dt$	x	$\int_0^x Y_0(t)dt$
0.00	−0.00000 00000	0.50	−0.56179 54559	1.00	−0.63706 93766	1.50	−0.51175 90340
.01	.03642 13434	.51	.56616 74822	.01	.63614 78909	.51	.50791 40388
.02	.06401 56299	.52	.57039 48062	.02	.63514 91483	.52	.50402 84689
.03	.08827 59463	.53	.57447 97923	.03	.63407 40058	.53	.50010 29784
.04	.11036 91138	.54	.57842 47218	.04	.63292 33121	.54	.49613 82195
0.05	−0.13084 88495	0.55	−0.58223 17979	1.05	−0.63169 79082	1.55	−0.49213 48426
.06	.15004 09780	.56	.58590 31498	.06	.63039 86275	.56	.48809 34956
.07	.16816 05644	.57	.58944 08377	.07	.62902 62965	.57	.48401 48249
.08	.18536 02867	.58	.59284 68562	.08	.62758 17343	.58	.47989 94748
.09	.20175 41601	.59	.59612 31379	.09	.62606 57537	.59	.47574 80876
0.10	−0.21743 05666	0.60	−0.59927 15570	1.10	−0.62447 91607	1.60	−0.47156 13039
.11	.23246 00155	.61	.60229 39327	.11	.62282 27552	.61	.46733 97623
.12	.24690 00554	.62	.60519 20316	.12	.62109 73311	.62	.46308 40994
.13	.26079 85379	.63	.60796 75708	.13	.61930 36765	.63	.45879 49501
.14	.27419 58694	.64	.61062 22206	.14	.61744 25737	.64	.45447 29476
0.15	−0.28712 66168	0.65	−0.61315 76067	1.15	−0.61551 47997	1.65	−0.45011 87230
.16	.29962 06829	.66	.61557 53123	.16	.61352 11261	.66	.44573 29056
.17	.31170 41857	.67	.61787 68807	.17	.61146 23197	.67	.44131 61232
.18	.32340 01314	.68	.62006 38170	.18	.60933 91420	.68	.43686 90014
.19	.33472 89362	.69	.62213 75901	.19	.60715 23502	.69	.43239 21642
0.20	−0.34570 88380	0.70	−0.62409 96341	1.20	−0.60490 26964	1.70	−0.42788 62338
.21	.35635 62260	.71	.62595 13505	.21	.60259 09287	.71	.42335 18307
.22	.36668 59066	.72	.62769 41094	.22	.60021 77905	.72	.41878 95734
.23	.37671 13214	.73	.62932 92513	.23	.59778 40213	.73	.41420 00788
.24	.38644 47270	.74	.63085 80879	.24	.59529 03564	.74	.40958 39619
0.25	−0.39589 73442	0.75	−0.63228 19041	1.25	−0.59273 75272	1.75	−0.40494 18362
.26	.40507 94841	.76	.63360 19587	.26	.59012 62613	.76	.40027 43129
.27	.41400 06541	.77	.63481 94861	.27	.58745 72825	.77	.39558 20020
.28	.42266 96485	.78	.63593 56968	.28	.58473 13112	.78	.39086 55112
.29	.43109 46258	.79	.63695 17790	.29	.58194 90642	.79	.38612 54469
0.30	−0.43928 31758	0.80	−0.63786 88991	1.30	−0.57911 12548	1.80	−0.38136 24134
.31	.44724 23770	.81	.63868 82032	.31	.57621 85932	.81	.37657 70132
.32	.45497 88470	.82	.63941 08175	.32	.57327 17863	.82	.37176 98471
.33	.46249 87870	.83	.64003 78493	.33	.57027 15377	.83	.36694 15143
.34	.46980 80196	.84	.64057 03880	.34	.56721 85483	.84	.36209 26117
0.35	−0.47691 20239	0.85	−0.64100 95057	1.35	−0.56411 35158	1.85	−0.35722 37348
.36	.48381 59646	.86	.64135 62577	.36	.56095 71350	.86	.35233 54771
.37	.49052 47197	.87	.64161 16837	.37	.55775 00980	.87	.34742 84304
.38	.49704 29038	.88	.64177 68081	.38	.55449 30942	.88	.34250 31845
.39	.50337 48900	.89	.64185 26406	.39	.55118 68100	.89	.33756 03274
0.40	−0.50952 48283	0.90	−0.64184 01770	1.40	−0.54783 19295	1.90	−0.33260 04453
.41	.51549 66636	.91	.64174 03997	.41	.54442 91341	.91	.32762 41224
.42	.52129 41508	.92	.64155 42781	.42	.54097 91027	.92	.32263 19412
.43	.52692 08690	.93	.64128 27694	.43	.53748 25118	.93	.31762 44822
.44	.53238 02340	.94	.64092 68186	.44	.53394 00353	.94	.31260 23240
0.45	−0.53767 55102	0.95	−0.64048 73595	1.45	−0.53035 23449	1.95	−0.30756 60433
.46	.54280 98209	.96	.63996 53149	.46	.52672 01101	.96	.30251 62148
.47	.54778 61582	.97	.63936 15971	.47	.52304 39978	.97	.29745 34114
.48	.55260 73911	.98	.63867 71081	.48	.51932 46729	.98	.29237 82039
.49	.55727 62746	.99	.63791 27403	.49	.51556 27982	.99	.28729 11611
0.50	−0.56179 54559	1.00	−0.63706 93766	1.50	−0.51175 90340	2.00	−0.28219 28501

TABLE OF THE INTEGRAL $\int_0^x Y_0(t)dt$—*Continued*

x	$\int_0^x Y_0(t)dt$	x	$\int_0^x Y_0(t)dt$	x	$\int_0^x Y_0(t)dt$	x	$\int_0^x Y_0(t)dt$
2.00	−0.28219 28501	2.50	−0.02480 29261	3.00	+0.19765 82565	3.50	+0.34090 94657
.01	.27708 38356	.51	.01982 95916	.01	.20141 04782	.51	.34277 91639
.02	.27196 46807	.52	.01487 12868	.02	.20512 99662	.52	.34460 77737
.03	.26683 59460	.53	.00992 84470	.03	.20881 64573	.53	.34639 52294
.04	.26169 81904	.54	.00500 15044	.04	.21246 96924	.54	.34814 14689
2.05	−0.25655 19707	2.55	−0.00009 08880	3.05	+0.21608 94163	3.55	+0.34984 64343
.06	.25139 78415	.56	+0.00480 29760	.06	.21967 53775	.56	.35151 00717
.07	.24623 63554	.57	.00967 96646	.07	.22322 73282	.57	.35313 23310
.08	.24106 80627	.58	.01453 87580	.08	.22674 50246	.58	.35471 31660
.09	.23589 35119	.59	.01937 98395	.09	.23022 82268	.59	.35625 25348
2.10	−0.23071 32490	2.60	+0.02420 24953	3.10	+0.23367 66086	3.60	+0.35775 03989
.11	.22552 78181	.61	.02900 63150	.11	.23709 02078	.61	.35920 67242
.12	.22033 77610	.62	.03379 08913	.12	.24046 85260	.62	.36062 14803
.13	.21514 36172	.63	.03855 58202	.13	.24381 14287	.63	.36199 46408
.14	.20994 59241	.64	.04330 07007	.14	.24711 86953	.64	.36332 61830
2.15	−0.20474 52170	2.65	+0.04802 51352	3.15	+0.25039 01091	3.65	+0.36461 60885
.16	.19954 20286	.66	.05272 87293	.16	.25362 54572	.66	.36586 43423
.17	.19433 68895	.67	.05741 10919	.17	.25682 45309	.67	.36707 09337
.18	.18913 03281	.68	.06207 18352	.18	.25998 71251	.68	.36823 58557
.19	.18392 28704	.69	.06671 05746	.19	.26311 30387	.69	.36935 91053
2.20	−0.17871 50399	2.70	+0.07132 69288	3.20	+0.26620 20748	3.70	+0.37044 06831
.21	.17350 73580	.71	.07592 05201	.21	.26925 40400	.71	.37148 05938
.22	.16830 03437	.72	.08049 09740	.22	.27226 87452	.72	.37247 88458
.23	.16309 45133	.73	.08503 79192	.23	.27524 60051	.73	.37343 54515
.24	.15789 03810	.74	.08956 09880	.24	.27818 56383	.74	.37435 04269
2.25	−0.15268 84586	2.75	+0.09405 98162	3.25	+0.28108 74674	3.75	+0.37522 37920
.26	.14748 92551	.76	.09853 40427	.26	.28395 13192	.76	.37605 55706
.27	.14229 32774	.77	.10298 33102	.27	.28677 70240	.77	.37684 57901
.28	.13710 10296	.78	.10740 72646	.28	.28956 44165	.78	.37759 44819
.29	.13191 30136	.79	.11180 55553	.29	.29231 33351	.79	.37830 16812
2.30	−0.12672 97284	2.80	+0.11617 78353	3.30	+0.29502 36222	3.80	+0.37896 74266
.31	.12155 16708	.81	.12052 37611	.31	.29769 51244	.81	.37959 17609
.32	.11637 93347	.82	.12484 29925	.32	.30032 76921	.82	.38017 47304
.33	.11121 32117	.83	.12913 51933	.33	.30292 11797	.83	.38071 63851
.34	.10605 37905	.84	.13340 00303	.34	.30547 54455	.84	.38121 67789
2.35	−0.10090 15573	2.85	+0.13763 71742	3.35	+0.30799 03520	3.85	+0.38167 59691
.36	.09575 69958	.86	.14184 62992	.36	.31046 57656	.86	.38209 40171
.37	.09062 05867	.87	.14602 70832	.37	.31290 15566	.87	.38247 09875
.38	.08549 28082	.88	.15017 92074	.38	.31529 75994	.88	.38280 69489
.39	.08037 41357	.89	.15430 23570	.39	.31765 37723	.89	.38310 19735
2.40	−0.07526 50420	2.90	+0.15839 62206	3.40	+0.31996 99576	3.90	+0.38335 61369
.41	.07016 59969	.91	.16246 04904	.41	.32224 60418	.91	.38356 95187
.42	.06507 74677	.92	.16649 48626	.42	.32448 19151	.92	.38374 22017
.43	.05999 99186	.93	.17049 90366	.43	.32667 74718	.93	.38387 42727
.44	.05493 38111	.94	.17447 27159	.44	.32883 26103	.94	.38396 58216
2.45	−0.04987 96041	2.95	+0.17841 56075	3.45	+0.33094 72328	3.95	+0.38401 69422
.46	.04483 77532	.96	.18232 74221	.46	.33302 12457	.96	.38402 77319
.47	.03980 87114	.97	.18620 78742	.47	.33505 45592	.97	.38399 82912
.48	.03479 29287	.98	.19005 66821	.48	.33704 70875	.98	.38392 87246
.49	.02979 08522	.99	.19387 35676	.49	.33899 87489	.99	.38381 91397
2.50	−0.02480 29261	3.00	+0.19765 82565	3.50	+0.34090 94657	4.00	+0.38366 96479

TABLE OF THE INTEGRAL $\int_0^x Y_0(t)dt$—*Continued*

x	$\int_0^x Y_0(t)dt$	x	$\int_0^x Y_0(t)dt$	x	$\int_0^x Y_0(t)dt$	x	$\int_0^x Y_0(t)dt$
4.00	+0.38366 96479	4.50	+0.32872 87513	5.00	+0.19971 93876	5.50	+0.03413 05806
.01	.38348 03637	.51	.32676 66951	.01	.19662 68746	.51	.03073 70184
.02	.38325 14054	.52	.32477 48020	.02	.19351 99138	.52	.02734 61667
.03	.38298 28946	.53	.32275 33371	.03	.19039 88443	.53	.02395 83593
.04	.38267 49561	.54	.32070 25675	.04	.18726 40060	.54	.02057 39293
4.05	+0.38232 77183	4.55	+0.31862 27627	5.05	+0.18411 57395	5.55	+0.01719 32086
.06	.38194 13130	.56	.31651 41943	.06	.18095 43857	.56	.01381 65284
.07	.38151 58752	.57	.31437 71359	.07	.17778 02865	.57	.01044 42187
.08	.38105 15432	.58	.31221 18633	.08	.17459 37839	.58	.00707 66086
.09	.38054 84589	.59	.31001 86544	.09	.17139 52209	.59	.00371 40262
4.10	+0.38000 67672	4.60	+0.30779 77892	5.10	+0.16818 49405	5.60	+0.00035 67983
.11	.37942 66164	.61	.30554 95497	.11	.16496 32865	.61	−0.00299 47490
.12	.37880 81579	.62	.30327 42197	.12	.16173 06030	.62	.00634 02912
.13	.37815 15466	.63	.30097 20853	.13	.15848 72345	.63	.00967 95046
.14	.37745 69403	.64	.29864 34343	.14	.15523 35259	.64	.01301 20668
4.15	+0.37672 45004	4.65	+0.29628 85564	5.15	+0.15196 98222	5.65	−0.01633 76568
.16	.37595 43910	.66	.29390 77432	.16	.14869 64690	.66	.01965 59545
.17	.37514 67796	.67	.29150 12884	.17	.14541 38120	.67	.02296 66414
.18	.37430 18369	.68	.28906 94870	.18	.14212 21971	.68	.02626 94000
.19	.37341 97365	.69	.28661 26363	.19	.13882 19705	.69	.02956 39143
4.20	+0.37250 06552	4.70	+0.28413 10351	5.20	+0.13551 34784	5.70	−0.03284 98697
.21	.37154 47729	.71	.28162 49839	.21	.13219 70673	.71	.03612 69529
.22	.37055 22725	.72	.27909 47849	.22	.12887 30836	.72	.03939 48520
.23	.36952 33398	.73	.27654 07421	.23	.12554 18741	.73	.04265 32566
.24	.36845 81638	.74	.27396 31610	.24	.12220 37853	.74	.04590 18576
4.25	+0.36735 69363	4.75	+0.27136 23488	5.25	+0.11885 91637	5.75	−0.04914 03476
.26	.36621 98522	.76	.26873 86141	.26	.11550 83561	.76	.05236 84206
.27	.36504 71092	.77	.26609 22673	.27	.11215 17088	.77	.05558 57722
.28	.36383 89079	.78	.26342 36200	.28	.10878 95685	.78	.05879 20995
.29	.36259 54519	.79	.26073 29855	.29	.10542 22812	.79	.06198 71011
4.30	+0.36131 69475	4.80	+0.25802 06786	5.30	+0.10205 01932	5.80	−0.06517 04775
.31	.36000 36040	.81	.25528 70153	.31	.09867 36505	.81	.06834 19307
.32	.35865 56334	.82	.25253 23130	.32	.09529 29986	.82	.07150 11642
.33	.35727 32504	.83	.24975 68906	.33	.09190 85832	.83	.07464 78834
.34	.35585 66727	.84	.24696 10683	.34	.08852 07494	.84	.07778 17954
4.35	+0.35440 61205	4.85	+0.24414 51675	5.35	+0.08512 98421	5.85	−0.08090 26091
.36	.35292 18167	.86	.24130 95110	.36	.08173 62057	.86	.08401 00349
.37	.35140 39872	.87	.23845 44225	.37	.07834 01844	.87	.08710 37852
.38	.34985 28601	.88	.23558 02272	.38	.07494 21220	.88	.09018 35743
.39	.34826 86664	.89	.23268 72515	.39	.07154 23617	.89	.09324 91182
4.40	+0.34665 16398	4.90	+0.22977 58227	5.40	+0.06814 12463	5.90	−0.09630 01348
.41	.34500 20163	.91	.22684 62693	.41	.06473 91181	.91	.09933 63437
.42	.34332 00347	.92	.22389 89209	.42	.06133 63190	.92	.10235 74668
.43	.34160 59361	.93	.22093 41081	.43	.05793 31900	.93	.10536 32277
.44	.33985 99642	.94	.21795 21625	.44	.05453 00718	.94	.10835 33520
4.45	+0.33808 23653	4.95	+0.21495 34166	5.45	+0.05112 73044	5.95	−0.11132 75671
.46	.33627 33880	.96	.21193 82040	.46	.04772 52271	.96	.11428 56027
.47	.33443 32834	.97	.20890 68591	.47	.04432 41786	.97	.11722 71905
.48	.33256 23048	.98	.20585 97172	.48	.04092 44967	.98	.12015 20639
.49	.33066 07080	.99	.20279 71144	.49	.03752 65185	.99	.12305 99588
4.50	+0.32872 87513	5.00	+0.19971 93876	5.50	+0.03413 05806	6.00	−0.12595 06129

TABLE OF THE INTEGRAL $\int_0^x Y_0(t)\,dt$—*Continued*

x	$\int_0^x Y_0(t)\,dt$	x	$\int_0^x Y_0(t)\,dt$	x	$\int_0^x Y_0(t)\,dt$	x	$\int_0^x Y_0(t)\,dt$
6.00	−0.12595 06129	6.50	−0.24338 05692	7.00	−0.29377 44843	7.50	−0.27002 13202
.01	.12882 37661	.51	.24509 92672	.01	.29401 88514	.51	.26883 52563
.02	.13167 91605	.52	.24679 04268	.02	.29423 29708	.52	.26762 34328
.03	.13451 65403	.53	.24845 39210	.03	.29441 68641	.53	.26638 60050
.04	.13733 56519	.54	.25008 96262	.04	.29457 05560	.54	.26512 31307
6.05	−0.14013 62438	6.55	−0.25169 74213	7.05	−0.29469 40741	7.55	−0.26383 49696
.06	.14291 80669	.56	.25327 71882	.06	.29478 74487	.56	.26252 16842
.07	.14568 08740	.57	.25482 88119	.07	.29485 07133	.57	.26118 34387
.08	.14842 44206	.58	.25635 21801	.08	.29488 39041	.58	.25982 04000
.09	.15114 84642	.59	.25784 71836	.09	.29488 70603	.59	.25843 27370
6.10	−0.15385 27646	6.60	−0.25931 37161	7.10	−0.29486 02239	7.60	−0.25702 06208
.11	.15653 70839	.61	.26075 16740	.11	.29480 34399	.61	.25558 42247
.12	.15920 11867	.62	.26216 09570	.12	.29471 67560	.62	.25412 37240
.13	.16184 48398	.63	.26354 14676	.13	.29460 02229	.63	.25263 92965
.14	.16446 78125	.64	.26489 31111	.14	.29445 38942	.64	.25113 11219
6.15	−0.16706 98762	6.65	−0.26621 57961	7.15	−0.29427 78261	7.65	−0.24959 93818
.16	.16965 08051	.66	.26750 94339	.16	.29407 20778	.66	.24804 42602
.17	.17221 03756	.67	.26877 39389	.17	.29383 67114	.67	.24646 59431
.18	.17474 83664	.68	.27000 92283	.18	.29357 17918	.68	.24486 46184
.19	.17726 45591	.69	.27121 52224	.19	.29327 73864	.69	.24324 04760
6.20	−0.17975 87372	6.70	−0.27239 18447	7.20	−0.29295 35658	7.70	−0.24159 37080
.21	.18223 06873	.71	.27353 90212	.21	.29260 04032	.71	.23992 45083
.22	.18468 01979	.72	.27465 66813	.22	.29221 79745	.72	.23823 30729
.23	.18710 70606	.73	.27574 47572	.23	.29180 63585	.73	.23651 95995
.24	.18951 10690	.74	.27680 31842	.24	.29136 56368	.74	.23478 42879
6.25	−0.19189 20197	6.75	−0.27783 19005	7.25	−0.29089 58935	7.75	−0.23302 73397
.26	.19424 97117	.76	.27883 08472	.26	.29039 72156	.76	.23124 89585
.27	.19658 39465	.77	.27979 99686	.27	.28986 96929	.77	.22944 93495
.28	.19889 45282	.78	.28073 92120	.28	.28931 34176	.78	.22762 87200
.29	.20118 12638	.79	.28164 85275	.29	.28872 84850	.79	.22578 72788
6.30	−0.20344 39625	6.80	−0.28252 78684	7.30	−0.28811 49927	7.80	−0.22392 52368
.31	.20568 24366	.81	.28337 71908	.31	.28747 30412	.81	.22204 28065
.32	.20789 65006	.82	.28419 64540	.32	.28680 27335	.82	.22014 02020
.33	.21008 59720	.83	.28498 56201	.33	.28610 41755	.83	.21821 76393
.34	.21225 06709	.84	.28574 46545	.34	.28537 74754	.84	.21627 53360
6.35	−0.21439 04200	6.85	−0.28647 35253	7.35	−0.28462 27441	7.85	−0.21431 35114
.36	.21650 50449	.86	.28717 22036	.36	.28384 00954	.86	.21233 23865
.37	.21859 43737	.87	.28784 06638	.37	.28302 96452	.87	.21033 21837
.38	.22065 82373	.88	.28847 88829	.38	.28219 15124	.88	.20831 31272
.39	.22269 64696	.89	.28908 68412	.39	.28132 58181	.89	.20627 54427
6.40	−0.22470 89068	6.90	−0.28966 45218	7.40	−0.28043 26862	7.90	−0.20421 93575
.41	.22669 53883	.91	.29021 19109	.41	.27951 22430	.91	.20214 51003
.42	.22865 57561	.92	.29072 89976	.42	.27856 46174	.92	.20005 29013
.43	.23058 98549	.93	.29121 57741	.43	.27758 99406	.93	.19794 29924
.44	.23249 75323	.94	.29167 22354	.44	.27658 83466	.94	.19581 56067
6.45	−0.23437 86388	6.95	−0.29209 83796	7.45	−0.27555 99715	7.95	−0.19367 09789
.46	.23623 30275	.96	.29249 42078	.46	.27450 49541	.96	.19150 93448
.47	.23806 05546	.97	.29285 97239	.47	.27342 34355	.97	.18933 09420
.48	.23986 10789	.98	.29319 49349	.48	.27231 55593	.98	.18713 60092
.49	.24163 44622	.99	.29349 98508	.49	.27118 14714	.99	.18492 47864
6.50	−0.24338 05692	7.00	−0.29377 44843	7.50	−0.27002 13202	8.00	−0.18269 75150

TABLE OF THE INTEGRAL $\int_0^x Y_0(t)\,dt$—*Concluded*

x	$\int_0^x Y_0(t)dt$	x	$\int_0^x Y_0(t)dt$	x	$\int_0^x Y_0(t)dt$	x	$\int_0^x Y_0(t)dt$
8.00	−0.18269 75150	8.50	−0.05650 66385	9.00	+0.07625 79635	9.50	+0.18361 20962
.01	.18045 44377	.51	.05380 33245	.01	.07875 20751	.51	.18531 40185
.02	.17819 57984	.52	.05109 76669	.02	.08123 55176	.52	.18699 54744
.03	.17592 18421	.53	.04838 99389	.03	.08370 80547	.53	.18865 63171
.04	.17363 28152	.54	.04568 04134	.04	.08616 94514	.54	.19029 64024
8.05	−0.17132 89652	8.55	−0.04296 93634	9.05	+0.08861 94739	9.55	+0.19191 55880
.06	.16901 05409	.56	.04025 70616	.06	.09105 78900	.56	.19351 37341
.07	.16667 77918	.57	.03754 37805	.07	.09348 44688	.57	.19509 07028
.08	.16433 09690	.58	.03482 97924	.08	.09589 89808	.58	.19664 63588
.09	.16197 03243	.59	.03211 53695	.09	.09830 11979	.59	.19818 05687
8.10	−0.15959 61109	8.60	−0.02940 07834	9.10	+0.10069 08937	9.60	+0.19969 32017
.11	.15720 85826	.61	.02668 63057	.11	.10306 78430	.61	.20118 41289
.12	.15480 79946	.62	.02397 22075	.12	.10543 18223	.62	.20265 32240
.13	.15239 46027	.63	.02125 87596	.13	.10778 26095	.63	.20410 03629
.14	.14996 86641	.64	.01854 62326	.14	.11011 99841	.64	.20552 54236
8.15	−0.14753 04364	8.65	−0.01583 48963	9.15	+0.11244 37272	9.65	+0.20692 82867
.16	.14508 01785	.66	.01312 50205	.16	.11475 36214	.66	.20830 88349
.17	.14261 81500	.67	.01041 68741	.17	.11704 94511	.67	.20966 69534
.18	.14014 46113	.68	.00771 07260	.18	.11933 10020	.68	.21100 25297
.19	.13765 98238	.69	.00500 68443	.19	.12159 80617	.69	.21231 54534
8.20	−0.13516 40494	8.70	−0.00230 54965	9.20	+0.12385 04194	9.70	+0.21360 56169
.21	.13265 75509	.71	+0.00039 30502	.21	.12608 78660	.71	.21487 29145
.22	.13014 05921	.72	.00308 85294	.22	.12831 01940	.72	.21611 72432
.23	.12761 34370	.73	.00578 06752	.23	.13051 71978	.73	.21733 85023
.24	.12507 63507	.74	.00846 92223	.24	.13270 86733	.74	.21853 65933
8.25	−0.12252 95988	8.75	+0.01115 39062	9.25	+0.13488 44183	9.75	+0.21971 14204
.26	.11997 34475	.76	.01383 44630	.26	.13704 42323	.76	.22086 28899
.27	.11740 81636	.77	.01651 06295	.27	.13918 79168	.77	.22199 09108
.28	.11483 40145	.78	.01918 21432	.28	.14131 52749	.78	.22309 53942
.29	.11225 12683	.79	.02184 87425	.29	.14342 61114	.79	.22417 62539
8.30	−0.10966 01934	8.80	+0.02451 01664	9.30	+0.14552 02334	9.80	+0.22523 34059
.31	.10706 10588	.81	.02716 61548	.31	.14759 74493	.81	.22626 67688
.32	.10445 41339	.82	.02981 64484	.32	.14965 75699	.82	.22727 62636
.33	.10183 96887	.83	.03246 07888	.33	.15170 04074	.83	.22826 18136
.34	.09921 79934	.84	.03509 89186	.34	.15372 57763	.84	.22922 33447
8.35	−0.09658 93188	8.85	+0.03773 05809	9.35	+0.15573 34929	9.85	+0.23016 07852
.36	.09395 39358	.86	.04035 55202	.36	.15772 33753	.86	.23107 40659
.37	.09131 21161	.87	.04297 34817	.37	.15969 52438	.87	.23196 31200
.38	.08866 41311	.88	.04558 42116	.38	.16164 89205	.88	.23282 78832
.39	.08601 02530	.89	.04818 74570	.39	.16358 42294	.89	.23366 82936
8.40	−0.08335 07540	8.90	+0.05078 29664	9.40	+0.16550 09969	9.90	+0.23448 42919
.41	.08068 59065	.91	.05337 04889	.41	.16739 90509	.91	.23527 58212
.42	.07801 59833	.92	.05594 97749	.42	.16927 82217	.92	.23604 28270
.43	.07534 12572	.93	.05852 05758	.43	.17113 83415	.93	.23678 52574
.44	.07266 20012	.94	.06108 26443	.44	.17297 92446	.94	.23750 30630
8.45	−0.06997 84885	8.95	+0.06363 57341	9.45	+0.17480 07674	9.95	+0.23819 61967
.46	.06729 09923	.96	.06617 95999	.46	.17660 27483	.96	.23886 46141
.47	.06459 97858	.97	.06871 39979	.47	.17838 50279	.97	.23950 82732
.48	.06190 51424	.98	.07123 86853	.48	.18014 74489	.98	.24012 71346
.49	.05920 73355	.99	.07375 34206	.49	.18188 98560	.99	.24072 11611
8.50	−0.05650 66385	9.00	+0.07625 79635	9.50	+0.18361 20962	10.00	+0.24129 03183

THE FLOW OF A COMPRESSIBLE VISCOUS FLUID THROUGH A STRAIGHT PIPE

By Y. H. Kuo

Introduction

The problem of determining a steady flow of incompressible viscous fluid through a straight pipe of any section is easily reduced to a problem of Dirichlet. This is the familiar Hagen-Poiseuille[1] flow, and a pipe of circular section is particularly well known. If, however, the fluid is compressible, the problem of steady flow becomes much more difficult. Although the problem has been treated experimentally[2] and by the rather crude method of hydraulics,[3] there does not appear to have been any previous mathematical treatment.

The method of the present paper is that of development in series in powers of the Mach number, which is supposed to be small. The operation with series is formal, no attempt being made to discuss the convergence.

The boundary condition of zero velocity on the wall of the tube, together with the information regarding the pressure drop, is not sufficient to make the mathematical problem definite. It is, in fact, a question of finding *a* solution rather than *the* solution; the simplest solution is taken, as in the classical case of the incompressible fluid, where the same partial indeterminacy also occurs.

Part I deals with a pipe of general section. The equations of motion expressed in terms of momentum vector and specific volume are given and the process of power series development is explained. This may, for small values of Mach number, be regarded as a process of successive approximations.

The zero approximation is the Hagen-Poiseuille flow.

In the first approximation the flow remains parallel to the walls of the pipe, and the determination of the velocity again reduces to a Dirichlet problem (5.3).

In the second approximation the velocity is no longer parallel to the walls. The component of momentum parallel to the walls is a linear function of the distance along the pipe (6.3), the determination of the coefficient of this function again depending on Dirichlet problems (6.8), (6.9). The transverse momentum component is proportional to $f_\alpha^{(2)}$ in (6.13), and this is independent of distance along the pipe; its determination is reduced to a Dirichlet problem (6.14) and a biharmonic problem (6.15).

In the zero approximation the pressure is a linear function of distance along the pipe (4.7), in the first approximation a quadratic function (5.8), and in the second approximation a cubic function (6.27).

The drag and flux are considered in §7.

In Part II the pipe is of circular section and explicit solutions are given.

[1] Hagen, Pogg. Ann., **46**, 423 (1839).

[2] Poiseuille, Comp. Rend. **11** and **12**, (1840–1).

[3] Reynolds, Phil. Trans. Roy. Soc. London, **174**, 935 (1883).

Part I. Pipe of a general section

1. The equations of motion

We shall consider a compressible viscous fluid flowing steadily through a pipe of any constant section, under the influence of the difference between the pressures applied at the ends of the pipe. It is assumed that the pipe has a well-rounded mouth-piece so that it introduces no initial disturbances. The domain of the mathematical theory is the interior of the pipe between two normal sections at a distance l' apart (the "length" of the pipe); these sections are called the "entrance" and "outlet".

Let x_i' be Cartesian coordinates with the origin at the mean centre of the entrance, the axis of x_0' lying along the axis of the pipe; let u_i' be the components of the velocity. Then in the pipe itself the motion satisfies the general equations[4,5] of steady motion in the absence of body force

$$\rho' u_j' u_{i,j}' = -p_{,i}' + \mu \Delta' u_i' + \frac{\mu}{3} \theta_{,i}', \tag{1.1}$$

where $\theta' = \partial u_j'/\partial x_j'$, $\Delta' (= \partial^2/\partial x_j' \partial x_j')$ is the Laplacian operator, ρ' the density, p' the pressure and μ the viscosity, which is a function of temperature, but within wide limits, independent of pressure. With these is associated the equation of continuity

$$(\rho' u_j')_{,j} = 0. \tag{1.2}$$

Here the equations are expressed in accordance with the indicial notation; the Latin suffixes have the range 0, 1, 2, the comma indicates partial differentiation and summation is understood for repeated suffixes. We shall assume the motion to be either isothermal or adiabatic, so that a definite pressure density relation exists. We shall in both cases treat μ as a constant, its variations under change of pressure being neglected in both cases and its variation under change of temperature being neglected in the adiabatic case.

We now find it convenient to define a few constants which are useful in the later calculation. Let $\bar{p}'$ be the mean pressure over the cross-section at the outlet, defined as

$$\bar{p}' = \frac{1}{A} \int\int p' \, dx_1' \, dx_2' \qquad \text{for} \quad x_0' = l', \tag{1.3}$$

where A is the area of the cross-section. Let $\bar{\rho}'$ be the density corresponding to $\bar{p}'$ according to the pressure-density relation of the fluid. Finally, let $\bar{u}'$ be the mean velocity over a cross-section, weighted by the density so that it is independent of the particular section chosen, namely,

$$\bar{u}' = \frac{1}{\bar{\rho}' A} \int\int \rho' u_0' \, dx_1' \, dx_2' \qquad \text{for any} \quad x_0'. \tag{1.4}$$

[4] W. Müller, Einführung in die Theorie der zähen Flüssigkeiten, (Leipzig, 1932) p. 13.
[5] H. Lamb, Hydrodynamics, (Cambridge, 1932) p. 577.

We shall now be able to express the equations in dimensionless form by the following transformation:

$$x_i = \frac{x_i'}{2m}, \qquad u_i = \frac{u_i'}{\bar{u}'}, \qquad p = \frac{p' - \bar{p}'}{\bar{\rho}'\bar{u}'^2}, \qquad \rho = \frac{\rho'}{\bar{\rho}'}, \tag{1.5}$$

where m is the hydraulic mean radius of the section, i.e. $m = A/P$ where P is the perimeter of the bounding curve C. Substituting (1.5) in (1.1) and (1.2), we have

$$\rho u_\beta u_{\alpha,\beta} + \rho u_0 u_{\alpha,0} = -p_{,\alpha} + \frac{2}{R}\Delta u_\alpha + \frac{2}{3R}\theta_{,\alpha}, \tag{1.6}$$

$$\rho u_\beta u_{0,\beta} + \rho u_0 u_{0,0} = -p_{,0} + \frac{2}{R}\Delta u_0 + \frac{2}{3R}\theta_{,0}, \tag{1.7}$$

$$(\rho u_\beta)_{,\beta} + (\rho u_0)_{,0} = 0, \tag{1.8}$$

the Greek suffixes having the range 1, 2; here $\theta = \partial u_j/\partial x_j$, R is the Reynolds' number, i.e., $R = 4m\bar{u}'/\nu$ where $\nu\left(= \frac{\mu}{\bar{\rho}'}\right)$ is the mean kinematic viscosity.

The boundary conditions to be satisfied by u_i are

$$u_i = 0 \quad \text{on} \quad C. \tag{1.9}$$

These seem to be the only essential boundary conditions. As we shall see later that, they are generally not sufficient to ensure a definite solution. We shall make use of this partial indeterminacy to obtain analytically simple solutions, hoping that the physical validity of such solutions may be justified in the same way that the solutions of Saint Venant in elasticity are justified.

It is convenient to have the constancy of the flux of mass across any section expressed in integral form; in dimensionless variables, it reads

$$\iint \rho u_0 \, dx_1 \, dx_2 = \frac{P^2}{4A}. \tag{1.10}$$

Once the velocity u_i is known we can solve for p from (1.6) and (1.7). As the differential equations involving p are of the first order there will be one arbitrary constant at our disposal, which will be determined by the condition at the outlet. This is

$$\iint p \, dx_1 \, dx_2 = 0 \qquad \text{for} \quad x_0 = l = \frac{l'}{2m}. \tag{1.11}$$

2. The pressure-density relation

The equations (1.6), (1.7) and (1.8) which involve five unknowns cannot be solved without using a relation connecting p and ρ. As remarked above, we shall suppose the change of state to be either isothermal or adiabatic. It would appear necessary to treat the case of a gas and that of a liquid separately. However, as we shall see below, it is possible to justify a common treatment as approximately valid from a physical point of view.

Let M denote the Mach number; it is defined as the ratio of the mean velocity $\bar{u}'$ of the fluid to that of sound in the fluid, namely,

$$M = \frac{\bar{u}'}{c_0}, \tag{2.1}$$

where c_0 is the velocity of sound in the fluid at the pressure $\bar{p}'$, i.e. $c_0^2 = (dp'/d\rho')_{\rho=\bar{\rho}'}$.

Suppose we are dealing with the perfect gas under the isothermal condition, the pressure-density relationship is Boyle's law

$$\rho' = kp',$$

where k is a constant. With this law the value of $\bar{\rho}'$ corresponding to $\bar{p}'$ is

$$\bar{\rho}' = k\bar{p}'.$$

Taking the difference of these two equations and converting to dimensionless variables, we have

$$\rho = 1 + M^2 p. \tag{2.2}$$

Similarly, under the adiabatic condition the pressure-density relationship takes the form

$$\rho^\gamma = 1 + \gamma M^2 p, \tag{2.3}$$

where γ is the ratio of the specific heats.

For mathematical reasons, we consider only those cases in which the viscosity μ may be regarded as a constant. This is certainly the case for the isothermal motion of a gas. It will not be accurate when the change is adiabatic, for the temperature no longer remains constant. However in gases, air for instance, the viscosity depends only slightly on temperature; for an increase of temperature from 0° to 20°C, the increase[6] in μ is about 6 percent, so we can take it as constant for small change in temperature.

In the case of a liquid, the general thermodynamic properties are rather vague and no definite laws have ever been established. However, it is an experimental fact that when[7] water flows through a pipe, the temperature everywhere is sensibly the same so long as the flow is laminar; and that[8] under a constant temperature, the pressure density-relation is approximately linear for a moderate range of pressures. Consequently, the isothermal law (2.2) will be assumed to be valid also for liquid.

As a mathematical convenience, we may regard the equation (2.2) as a special case of the equation (2.3), although they represent entirely different physical processes. Hence we shall use (2.3), reducing to (2.2) when required by putting $\gamma = 1$.

[6] H. Lamb, Loc. cit.[5] p. 576.
[7] Barnes and Coker, Proc. Roy. Soc. A 74, 341 (1904).
[8] P. W. Bridgman, Proc. Roy. Soc. A 48, 309 (1912); A 49, 1 (1913); A 66, 185 (1931).

3. Method of approximation

We have now in (1.6), (1.7), (1.8) and (2.3) a complete system of non-linear partial differential equations of the second order for the variables u_i, p, and ρ. We shall proceed towards their solution by a method of successive approximations in which the equations are linearized by suitable assumptions.

When we eliminate p between (1.6) and (1.7) and introduce the momentum vector V_i

$$u_i = \sigma V_i, \tag{3.1}$$

where $\sigma = \frac{1}{\rho}$, we obtain two equations

$$\{V_\beta(\sigma V_\alpha)_{,\beta} + V_0(\sigma V_\alpha)_{,0}\}_{,0} - \{V_\beta(\sigma V_0)_{,\beta} + V_0(\sigma V_0)_{,0}\}_{,\alpha} = \frac{2}{R}\Delta\{(\sigma V_\alpha)_{,0} - (\sigma V_0)_{,\alpha}\}\ ; \tag{3.2}$$

and the equation of continuity becomes

$$V_{\beta,\beta} + V_{0,0} = 0. \tag{3.3}$$

We suppose that the dependent variables can be developed in series of ascending power of M^2:

$$V_i = V_i^{(0)} + M^2 V_i^{(1)} + M^4 V_i^{(2)} + \cdots, \tag{3.4}$$

$$\sigma = 1 + M^2\sigma^{(1)} + M^4\sigma^{(2)} + \cdots, \tag{3.5}$$

$$p = p^{(0)} + M^2 p^{(1)} + M^4 p^{(2)} + \cdots. \tag{3.6}$$

By virtue of (2.3) which can be written as

$$\sigma = (1 + \gamma M^2 p)^{-\frac{1}{\gamma}}, \tag{3.7}$$

(3.5) and (3.6) give the following relations by binomial expansion:

$$\sigma^{(1)} = -p^{(0)}, \qquad \sigma^{(2)} = -p^{(1)} + \frac{\gamma+1}{2}(p^{(0)})^2,$$
$$\sigma^{(3)} = -p^{(2)} + (\gamma+1)p^{(1)}p^{(0)} - \frac{(\gamma+1)(2\gamma+1)}{6}(p^{(0)})^3, \cdots. \tag{3.8}$$

Substituting the expressions (3.4) and (3.5) in (3.2) and (3.3) and collecting terms of equal powers of M, we obtain sets of equations from the vanishing of the coefficients of all powers of M. These equations are complicated, but it is necessary to set them down in order to make the later developments intelligible. The equations corresponding to M^0 M^2, M^4 are as follows

$$\{V_\beta^{(0)} V_{\alpha,\beta}^{(0)} + V_0^{(0)} V_{\alpha,0}^{(0)}\}_{,0} - \{V_\beta^{(0)} V_{0,\beta}^{(0)} + V_0^{(0)} V_{0,0}^{(0)}\}_{,\alpha} = \frac{2}{R}\Delta\{V_{\alpha,0}^{(0)} - V_{0,\alpha}^{(0)}\}, \tag{3.9}$$

$$V^{(0)}_{\beta,\beta} + V^{(0)}_{0,0} = 0; \tag{3.10}$$

$$\begin{aligned}&\{V^{(0)}_\beta u^{(1)}_{\alpha,\beta} + V^{(1)}_\beta V^{(0)}_{\alpha,\beta} + V^{(0)}_0 u^{(1)}_{\alpha,0} + V^{(1)}_0 V^{(0)}_{\alpha,0}\}_{,0}\\ &- \{V^{(0)}_\beta u^{(1)}_{0,\beta} + V^{(1)}_\beta V^{(0)}_{0,\beta} + V^{(0)}_0 u^{(1)}_{0,0} + V^{(1)}_0 V^{(0)}_{0,0}\}_{,\alpha} = \frac{2}{R}\Delta\{u^{(1)}_{\alpha,0} - u^{(1)}_{0,\alpha}\},\end{aligned} \tag{3.11}$$

$$V^{(1)}_{\beta,\beta} + V^{(1)}_{0,0} = 0; \tag{3.12}$$

$$\begin{aligned}&\{V^{(0)}_\beta u^{(2)}_{\alpha,\beta} + V^{(1)}_\beta u^{(1)}_{\alpha,\beta} + V^{(2)}_\beta V^{(0)}_{\alpha,\beta} + V^{(0)}_0 u^{(2)}_{\alpha,0} + V^{(1)}_0 u^{(1)}_{\alpha,0} + V^{(2)}_0 V^{(0)}_{\alpha,0}\}_{,0}\\ &-\{V^{(0)}_\beta u^{(2)}_{0,\beta} + V^{(1)}_\beta u^{(1)}_{0,\beta} + V^{(2)}_\beta V^{(0)}_{0,\beta} + V^{(0)}_0 u^{(2)}_{0,0} + V^{(1)}_0 u^{(1)}_{0,0} + V^{(2)}_0 V^{(0)}_{0,0}\}_{,\alpha}\\ &= \frac{2}{R}\Delta\{u^{(2)}_{\alpha,0} - u^{(2)}_{0,\alpha}\},\end{aligned} \tag{3.13}$$

$$V^{(2)}_{\beta,\beta} + V^{(2)}_{0,0} = 0; \tag{3.14}$$

where

$$u^{(m)}_i = \sum_{n=0}^{n=m} \sigma^{(n)} V^{(m-n)}_i.$$

To solve for the pressure, we shall also require equations deduced by substituting from (3.4), (3.5), (3.6) in the equations of motion (1.6), (1.7), (1.8). These equations corresponding to M^0, M^2, M^4 are as follows:

$$V^{(0)}_j V^{(0)}_{i,j} = -p^{(0)}_{,i} + \frac{2}{R}\Delta V^{(0)}_i, \tag{3.15}$$

$$V^{(0)}_j u^{(1)}_{i,j} + V^{(1)}_j V^{(0)}_{i,j} = -p^{(1)}_{,i} + \frac{2}{R}\Delta u^{(1)}_i + \frac{2}{3R}\theta^{(1)}_{,i}, \tag{3.16}$$

$$V^{(0)}_j u^{(2)}_{i,j} + V^{(1)}_j u^{(1)}_{i,j} + V^{(2)}_j V^{(0)}_{i,j} = -p^{(2)}_{,i} + \frac{2}{R}\Delta u^{(2)}_i + \frac{2}{3R}\theta^{(2)}_{,i}, \tag{3.17}$$

where $\theta^{(m)} = u^{(m)}_{j,j}$.

We shall start to solve these sets of equations from zero approximation, for which a simple assumption will be made so that the equations are linearized. After the velocity for the zero approximation has been found, we can put it in the equations (3.17) and (3.18) and thereby the pressure $p^{(0)}$ can be calculated. This gives $\sigma^{(1)}$ through the first of (3.8). Then substituting $V^{(0)}_i$ and $\sigma^{(1)}$ in the equations (3.11) we can calculate the corresponding quantities for the next approximation; and so on. In this manner, we can proceed to any order of approximation as it is required.

The boundary conditions on C for the momentum components of all approximations are

$$V^{(n)}_0 = V^{(n)}_\alpha = 0. \qquad (n = 0, 1, 2, \cdots) \tag{3.18}$$

These, as we said before, are generally insufficient to determine all the arbitrary functions arising from the integration of the equations. We shall introduce further the "normalization conditions" which follow from (1.10), namely,

$$\iint V_0^{(0)}\, dx_1\, dx_2 = \frac{P^2}{4A}, \tag{3.19}$$

$$\iint V_0^{(n)}\, dx_1\, dx_2 = 0 \qquad (n = 1, 2, 3, \cdots) \tag{3.20}$$

for any x_0.

For the pressure, we have only the end conditions which require the constants in any case to be in conformity with (1.11) such that

$$\iint p^{(n)}\, dx_1\, dx_2 = 0 \qquad (n = 0, 1, 2, \cdots) \tag{3.21}$$

for $x_0 = l$.

4. Zero approximation

To solve the equations for the zero approximation, we assume $V_\alpha^{(0)} = 0$; then the equations (3.9) and (3.10) are reduced to

$$\Delta V_{0,\alpha}^{(0)} = 0, \tag{4.1}$$

$$V_{0,0}^{(0)} = 0. \tag{4.2}$$

These equations are satisfied by taking for $V_0^{(0)}$ any function of x_1, x_2, which satisfies

$$\underset{2}{\Delta} V_0^{(0)} = -\Pi, \tag{4.3}$$

where $\underset{2}{\Delta}$ is the two-dimensional Laplacian operator and Π is a constant. The solution is to satisfy the conditions

$$V_0^{(0)} = 0 \quad \text{on} \quad C, \tag{4.4}$$

$$\iint V_0^{(0)}\, dx_1\, dx_2 = \frac{P^2}{4A}. \tag{4.5}$$

On account of (4.5) from which the constant Π is determined, the solution is unique.

If we put $V_0^{(0)}$ back into (3.15), we get

$$\Pi = -\frac{R}{2}\, p_{,0}^{(0)}. \tag{4.6}$$

On integration of this equation, we obtain by (3.21)

$$p^{(0)} = \frac{2\Pi}{R}(l - x_0). \tag{4.7}$$

The zero approximation is the well-known Poiseuille motion.

Knowing $p^{(0)}$, we have, by the first of (3.8)

$$\sigma^{(1)} = -\frac{2\Pi}{R}(l - x_0). \tag{4.8}$$

5. First approximation

For the first approximation, we again assume that the components of the velocity, except that along x_0, vanish. Thus on putting $V_\alpha^{(0)} = 0$ the equations (3.11) and (3.12) are simplified to

$$\{V_0^{(0)}(V_0^{(1)} + \sigma^{(1)} V_0^{(0)})_{,0}\}_{,\alpha} = \frac{2}{R}\Delta V_{0,\alpha}^{(1)}, \tag{5.1}$$

$$V_{0,0}^{(1)} = 0. \tag{5.2}$$

These equations are satisfied by taking for $V_0^{(1)}$ any function of x_1, x_2 which satisfies

$$\underset{2}{\Delta} V_0^{(1)} = \Pi\{V_0^{(0)}\}^2 + C_0^{(1)}, \tag{5.3}$$

where $C_0^{(1)}$ is an arbitrary constant. The boundary condition

$$V_0^{(1)} = 0 \quad \text{on} \quad C, \tag{5.4}$$

together with the further condition

$$\iint V_0^{(1)}\, dx_1\, dx_2 = 0 \quad \text{for any} \quad x_0, \tag{5.5}$$

is sufficient for a unique solution.

With $V_0^{(1)}$ so calculated, we can proceed to solve for $p^{(1)}$ from the equations (3.16), namely,

$$0 = -p_{,\alpha}^{(1)} + \frac{2}{3R}(\sigma_{,0}^{(1)} V_0^{(0)})_{,\alpha}, \tag{5.6}$$

$$0 = -p_{,0}^{(1)} + \frac{2C_0^{(1)}}{R} + \frac{4\Pi^2}{R^2}(l - x_0). \tag{5.7}$$

Integrating we have

$$p^{(1)} = -\frac{2\Pi^2}{R^2}(l - x_0)^2 - \frac{2C_0^{(1)}}{R}(l - x_0) + \frac{4\Pi}{3R^2}(V_0^{(0)} - 1), \tag{5.8}$$

where the constants of integration are determined in accordance with (1.11). From the second of (3.8), we have

$$\sigma^{(2)} = \frac{2\Pi^2(\gamma + 2)}{R^2}(l - x_0)^2 + \frac{2C_0^{(1)}}{R}(l - x_0) - \frac{4\Pi}{3R^2}(V_0^{(0)} - 1), \tag{5.9}$$

6. Second approximation

Making use of the results obtained above, we reduce the equations (3.13) to

$$V_0^{(0)} V_{\alpha,00}^{(2)} - (V_\beta^{(2)} V_{0,\beta}^{(0)} + V_0^{(0)} V_{0,0}^{(2)})_{,\alpha} = \frac{2}{R}\Delta(V_{\alpha,0}^{(2)} - V_{0,\alpha}^{(2)}) + \frac{4(\gamma + 1)\Pi^2}{R^2} x_0\{V_0^{(0)}\}^2 + F_{,\alpha}, \tag{6.1}$$

where

$$F = \frac{4\Pi}{R} V_0^{(1)} V_0^{(0)} - \frac{2}{R}\left(C_0^{(1)} + \frac{2(\gamma+1)\Pi^2}{R} l\right)\{V_0^{(0)}\}^2 - \frac{2}{R}\left\{\frac{4(3\gamma+8)\Pi^2}{3R^2} V_0^{(0)} - \frac{8\pi}{3R^2} V_{0,\beta}^{(0)} V_{0,\beta}^{(0)}\right\},$$

while the equation of continuity remains

$$V_{\beta,\beta}^{(2)} + V_{0,0}^{(2)} = 0. \tag{6.2}$$

We shall seek a particular solution of the form

$$V_0^{(2)} = f_0^{(2)} + x_0 g_0^{(2)}, \tag{6.3}$$

$$V_\alpha^{(2)} = f_\alpha^{(2)}, \tag{6.4}$$

where $f_0^{(2)}$, $g_0^{(2)}$, and $f_\alpha^{(2)}$ are functions of x_1 and x_2. Substituting these in (6.1) and equating the coefficients of equal powers of x_0, we obtain the following set of equations:

$$\frac{2}{R}\underset{2}{\Delta} f_{0,\alpha}^{(2)} - (f_\beta^{(2)} V_{0,\beta}^{(0)} + V_0^{(0)} g_0^{(2)})_{,\alpha} = F_{,\alpha}, \tag{6.5}$$

$$0 = -\frac{2}{R}\underset{2}{\Delta} g_{0,\alpha}^{(2)} + \frac{4(\gamma+1)\Pi^2}{R^2}\{V_0^{(0)}\}_{,\alpha}^2. \tag{6.6}$$

Similarly, the equation of continuity (6.2) yields

$$f_{\beta,\beta}^{(2)} + g_0^{(2)} = 0. \tag{6.7}$$

The integration of equations (6.5) and (6.6) gives at once two equations:

$$\underset{2}{\Delta} g_0^{(2)} = \frac{2(\gamma+1)\Pi^2}{R}(V_0^{(0)})^2 + C_0^{(2)}, \tag{6.8}$$

$$\underset{2}{\Delta} f_0^{(2)} = \frac{R}{2}(F + f_\beta^{(2)} V_{0,\beta}^{(0)} + V_0^{(0)} g_0^{(2)}) + D_0^{(2)}, \tag{6.9}$$

where $C_0^{(2)}$ and $D_0^{(2)}$ are arbitrary constants. The equation (6.8) now involves only one unknown $g_0^{(2)}$ and is to be solved subject to the boundary condition

$$g_0^{(2)} = 0 \quad \text{on} \quad C, \tag{6.10}$$

and the further condition

$$\iint g_0^{(2)}\, dx_1\, dx_2 = 0. \tag{6.11}$$

We have now in (6.7) and (6.9) two equations for three unknowns. This problem is, at the first sight, indeterminate. However, by eliminating $p^{(2)}$ between the equations (3.17) we obtain another independent equation:

$$\underset{2}{\Delta}(V_{\alpha,\beta}^{(2)} - V_{\beta,\alpha}^{(2)}) = 0. \tag{6.12}$$

To solve this system of equations, we write, without loss of generality, for the vector $f_\alpha^{(2)}$:

$$f_\alpha^{(2)} = \varphi_{,\alpha}^{(2)} + \epsilon_{\alpha\beta}\psi_{,\beta}^{(2)}, \tag{6.13}$$

where $\varphi^{(2)}$ and $\psi^{(2)}$ are functions of x_1, x_2, and $\epsilon_{\alpha\beta}$ the permutation symbol ($\epsilon_{11} = \epsilon_{22} = 0$; $\epsilon_{12} = -\epsilon_{21} = 1$). Substituting (6.13) in (6.7) and (6.12) we have

$$\underset{2}{\Delta}\varphi^{(2)} + g_0^{(2)} = 0, \tag{6.14}$$

$$\underset{2}{\Delta}\underset{2}{\Delta}\psi^{(2)} = 0. \tag{6.15}$$

The boundary conditions for $\varphi^{(2)}$ and $\psi^{(2)}$ on C, according to (3.18), are

$$\frac{\partial\varphi^{(2)}}{\partial s} + \frac{\partial\psi^{(2)}}{\partial n} = 0, \tag{6.16}$$

$$\frac{\partial\varphi^{(2)}}{\partial n} - \frac{\partial\psi^{(2)}}{\partial s} = 0, \tag{6.17}$$

where n refers to the outer normal. Furthermore, we have from (6.11) on eliminating $g_0^{(2)}$ from (6.14)

$$\int_C \frac{\partial\varphi^{(2)}}{\partial n}\, ds = 0,$$

by Green's theorem. We shall satisfy this condition by the special assumption

$$\frac{\partial\varphi^{(2)}}{\partial n} = 0, \tag{6.18}$$

on C. On account of this, (6.17) and (6.16) now become

$$\psi^{(2)} = \text{const.} = 0, \tag{6.19}$$

$$\frac{\partial\psi^{(2)}}{\partial n} = -\frac{\partial\varphi^{(2)}}{\partial s}. \tag{6.20}$$

A substitution of $g_0^{(2)}$ and $f_\alpha^{(2)}$ in (6.9) will enable us to solve for $f_0^{(2)}$. Thus

$$\underset{2}{\Delta} f_0^{(2)} = \frac{R}{2}\left(F + f_\beta^{(2)} V_{0,\beta}^{(0)} + V_0^{(0)} g_0^{(2)}\right) + D_0^{(2)} \tag{6.21}$$

with the boundary condition

$$f_0^{(2)} = 0 \quad \text{on} \quad C, \tag{6.22}$$

and a further condition

$$\iint f_0^{(2)}\, dx_1\, dx_2 = 0. \tag{6.23}$$

It is seen that the equations (6.8), (6.14), (6.21) and (6.15) together with the boundary conditions (6.10), (6.18), (6.22) and (6.19) (6.20) constitute the complete mathematical problem. When these equations are solved, (6.3), (6.4) and (6.13) will give the momentum components.

The equations for $p^{(2)}$ now take the following forms:

$$0 = -p^{(2)}_{,\alpha} + \frac{2}{R}\Delta_2 f^{(2)}_\alpha + \frac{2}{3R}(\sigma^{(1)}_{,0} V^{(1)}_0 + \sigma^{(2)}_{,0} V^{(0)}_0)_{,\alpha}, \tag{6.24}$$

$$V^{(0)}_{0,\beta} V^{(2)}_\beta + V^{(0)}_0 g^{(2)}_0 + 2\sigma^{(1)}_{,0} V^{(1)}_0 V^{(0)}_0 + \sigma^{(2)}_{,0}(V^{(0)}_0)^2 = -p^{(2)}_{,0} + \frac{2}{R}\Delta(V^{(2)}_0 + \sigma^{(1)} V^{(1)}_0 + \sigma^{(2)} V^{(0)}_0) + \frac{2}{3R}\sigma^{(2)}_{,00} V^{(0)}_0. \tag{6.25}$$

A lengthy reduction gives

$$p^{(2)}_{,\alpha} = \frac{2}{R}\Delta_2 f^{(2)}_\alpha + \chi_{,\alpha} + P_0(l - x_0)V^{(0)}_{0,\alpha}, \tag{6.26}$$

$$p^{(2)}_{,0} = -P_0 V^{(0)}_0 - P_1 - 2P_2(l - x_0) - 3P_3(l - x_0)^2, \tag{6.27}$$

where

$$\chi = \frac{2}{3R}\left(\sigma^{(1)}_{,0} V^{(1)}_0 - \frac{2C^{(1)}_0}{R} V^{(0)}_0\right), \qquad P_0 = -\frac{8\Pi^2(\gamma + 2)}{3R^3}$$

$$P_1 = -\frac{2}{R}(D^{(2)}_0 + lC^{(2)}_0), \qquad P_2 = \frac{1}{R}\left(C^{(2)}_0 + \frac{4\Pi}{R} C^{(1)}_0\right), \qquad P_3 = \frac{4(\gamma + 2)\Pi^3}{3R^3}.$$

Integreting we have

$$p^{(2)} = P_0(l - x_0)V^{(0)}_0 + P_1(l - x_0) + P_2(l - x_0)^2 + P_3(l - x_0)^3 + \chi + \frac{4C^{(1)}_0}{3R^2}. \tag{6.27}$$

From the third of (3.8), we have

$$\sigma^{(2)} = -\frac{4\Pi^3(2\gamma + 3)(\gamma + 2)}{3R^3}(l - x_0)^3 - \left\{\frac{4\Pi C^{(1)}_0(\gamma + 2)}{R^2} + \frac{C^{(2)}_0}{R}\right\}(l - x_0)^2 + \frac{8\Pi^2(\gamma + 2)}{3R^3}(l - x_0) + \frac{8\Pi^2}{3R^3}(V^{(0)}_0 + \gamma + 1) - \chi - \frac{4C^{(1)}_0}{3R^2}. \tag{6.28}$$

7. Drag and flux

In steady flow along the pipe, the skin friction is given by

$$\tau = -\mu\left(\frac{\partial u'_0}{\partial n'}\right)_{n'=0},$$

where n' refers to outer normal to the boundary C. The total resistance or drag is, therefore,

$$D = -2m\mu\bar{u}' \iint \left(\frac{\partial u_0}{\partial n}\right)_{n=0} dS,$$

where the integral is extended over the whole inner surface of the pipe. Introducing the drag coefficient $C_D = D/\frac{1}{2}\bar{\rho}'\bar{u}'^2S'$, and expressing in terms of V_0, we find

$$C_D = -\frac{4}{\bar{R}\bar{S}}\iint\left\{\frac{\partial}{\partial n}(\sigma V_0)\right\}_{n=0} dS. \tag{7.1}$$

Now both V_0 and σ have already been expanded in powers of M. Substituting V_0 and σ respectively from (3.4) and (3.5) in (7.1), we get a new series

$$C_D = C_D^{(0)} + M^2C_D^{(1)} + M^4C_D^{(2)} + \cdots \tag{7.2}$$

of which the general term is

$$C_D^{(r)} = -\frac{4}{\bar{R}\bar{S}}\iint\left\{\sum_{s=0}^{s=r}\frac{\partial}{\partial n}(\sigma^{(s)}V_0^{(r-s)})\right\}_{n=0} dS. \tag{7.3}$$

Another important quantity in the problem of pipe-flow is the flux across any section of the pipe which is given by

$$Q = \iint \rho' u_0' dx_1' dx_2'.$$

From (3.19) and (3.20) it follows that

$$Q = \bar{\rho}'\,\bar{u}'\,A. \tag{7.4}$$

Let H denote the pressure-head. It is easy to show that

$$H = \frac{4m^2\bar{\rho}'\bar{u}'^2}{A}\iint (p)_{x_0=0}\, dx_1\, dx_2. \tag{7.5}$$

This quantity, aside from the dimensions of the pipe, is the only parameter on which the nature of the flow depends. Consequently, it is convenient to define a flux coefficient C_Q such that

$$\frac{Q}{H\cdot\dfrac{Al'}{p}} = \frac{1}{C_Q}, \tag{7.6}$$

where

$$C_Q = \frac{4m^2lR}{A}\iint (p)_{x_0=0}\, dx_1\, dx_2.$$

Substituting p from (3.6), we will have also a series

$$C_Q = C_Q^{(0)} + M^2C_Q^{(1)} + M^4C_Q^{(2)} + \cdots, \tag{7.7}$$

where the rth term is

$$C_Q^{(r)} = \frac{4m^2lR}{A}\iint (p^{(r)})_{x_0=0}\, dx_1\, x_2. \tag{7.8}$$

If $p^{(r)}$ is known, $C_Q^{(r)}$ can be calculated; and, when put back in (7.7), we will have C_Q

The way in which the flux coefficient is defined here is not conventional. It is simply a mathematical convenience. For actual calculation its reciprocal should be considered.

Part II. Flow through a circular pipe

8. Poiseuille motion

In this section, we shall choose, as an example to illustrate our method, a pipe with circular section of radius a. We shall use cylindrical coordinates, and assume that the motion possesses axial symmetry. Taking the origin at the centre of the section, the relations between the two systems of coordinates are:

$$x_0 = x_0,$$

$$x_1 = r\cos\theta,$$

$$x_2 = r\sin\theta,$$

where r and θ are the polar coordinates in the x_1 and x_2 plane and r is dimensionless, defined according to (1.5) with $m = a/2$.

As each dependent variable is now independent of θ, the Laplacian operator may be written

$$\underset{2}{\Delta} = \frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right). \tag{8.1}$$

Consequently, equation (4.3) becomes

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dV_0^{(0)}}{dr}\right) = -\Pi. \tag{8.2}$$

With boundary condition (4.4), this gives

$$V_0^{(0)} = \frac{\Pi}{4}(1 - r^2). \tag{8.3}$$

The constant Π is to be determined in accordance with the condition (4.5); this yields

$$\Pi = 8. \tag{8.4}$$

Thus the velocity distribution for Poiseuille motion assumes the usual form:

$$V_0^{(0)} = 2(1 - r^2), \qquad V_a^{(0)} = 0. \tag{8.5}$$

The corresponding pressure is

$$p^{(0)} = \frac{16}{R}(l - x_0), \tag{8.6}$$

where the Reynolds number now is $R = 2a\tilde{u}'/\nu$. Similarly, we have

$$\sigma^{(1)} = -\frac{16}{R}(l - x_0). \tag{8.7}$$

9. First approximation

For the first approximation, we have made the assumption that the component of momentum in the x_1, x_2 plane is zero. The component along x_0 is given by equation (5.3), namely

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dV_0^{(1)}}{dr}\right) = 32(1 - r^2)^2 + C_0^{(1)}. \tag{9.1}$$

Integrating we have

$$V_0^{(1)} = 32\left(\frac{r^2}{4} - \frac{r^4}{8} + \frac{r^6}{36}\right) + \frac{C_0^{(1)}}{4}r^2 + C_1^{(1)}. \tag{9.2}$$

The values of $C_0^{(1)}$, $C_1^{(1)}$ are found by substitution in (5.4), (5.5) and we obtain

$$C_0^{(1)} = -16, \tag{9.3}$$

$$C_1^{(1)} = -\frac{8}{9}, \tag{9.4}$$

and so

$$V_0^{(1)} = -\frac{4}{9}(-2r^6 + 9r^4 - 9r^2 + 2). \tag{9.5}$$

The pressure $p^{(1)}$ is then given by

$$p^{(1)} = -\frac{128}{R}(l - x_0)^2 + \frac{32}{R}(l - x_0) + \frac{32}{3R^2}(1 - 2r^2). \tag{9.6}$$

Consequently, the specific volume $\sigma^{(2)}$ is

$$\sigma^{(2)} = \frac{128(\gamma + 2)}{R}(l - x_0)^2 - \frac{32}{R}(l - x_0) - \frac{32}{3R^2}(1 - 2r^2). \tag{9.7}$$

10. Second approximation

To simplify the equations (6.1), we have made the assumption that the momentum component $V_0^{(2)}$ for the second approximation possesses the following form:

$$V_0^{(2)} = f_0^{(2)} + x_0 g_0^{(2)},$$

where $f_0^{(2)}$ and $g_0^{(2)}$ are both functions of r alone due to the symmetry. Now $g_0^{(2)}$ is found to satisfy

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dg_0^{(2)}}{dr}\right) = \frac{572(\gamma + 1)}{R}(1 - r^2)^2 + C_0^{(2)}; \tag{10.1}$$

which has the solution

$$g_0^{(2)} = \frac{512(\gamma+1)}{R}\left(\frac{r^2}{4} - \frac{r^4}{8} + \frac{r^6}{36}\right) + \frac{C_0^{(2)}}{4} r^2 + C_1^{(2)}. \tag{10.2}$$

The constants of integration, determined to satisfy the conditions (6.10), (6.11) are

$$C_0^{(2)} = -\frac{256(\gamma+1)}{R}, \tag{10.3}$$

$$C_1^{(2)} = -\frac{128(\gamma+1)}{9R}. \tag{10.4}$$

Substituting back in (10.2), we have

$$g_0^{(2)} = \frac{64(\gamma+1)}{R}(2r^6 - 9r^4 + 9r^2 - 2). \tag{10.5}$$

As the equation (6.21) for $f_0^{(2)}$ involves the momentum components in the r, θ plane, it cannot be solved so long as the latter quantities remain unknown. They were connected by $\varphi^{(2)}$ and $\psi^{(2)}$ in (6.13) which can be solved from (6.14) and (6.15), namely:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\varphi^{(2)}}{dr}\right) + g_0^{(2)} = 0, \tag{10.6}$$

$$\left\{\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right)\right\}\left\{\frac{1}{r}\frac{d}{dr}\left(r\frac{d\psi^{(2)}}{dr}\right)\right\} = 0, \tag{10.7}$$

with the following boundary conditions for $r = 1$:

$$\frac{d\varphi^{(2)}}{dr} = 0, \tag{10.8}$$

$$\psi^{(2)} = 0, \tag{10.9}$$

$$\frac{d\psi^{(2)}}{dr} = 0. \tag{10.10}$$

Substituting $g_0^{(2)}$ from (10.5) in (10.6), we find

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\varphi^{(2)}}{dr}\right) = -\frac{64(\gamma+1)}{9R}(2r^6 - 9r^4 + 9r^2 - 2). \tag{10.11}$$

The solution satisfying the boundary condition (10.8), is

$$\varphi^{(2)} = -\frac{2(\gamma+1)}{9R} r^8 - 8r^6 + 18r^4 - 16r^2) + \varphi_0, \tag{10.12}$$

where the constant of integration φ_0 may be arbitrary.

The solution of (10.7), with the boundary conditions (10.9), (10.10) is easily shown to be

$$\psi^{(2)} = \frac{k}{4}(2r^2 \log r - r^2 + 1), \tag{10.13}$$

the constant k being arbitrary. But the vorticity must be finite at the centre; this gives $k = 0$ and hence

$$\psi^{(2)} = 0. \tag{10.14}$$

Thus we arrive at the following results for the momentum components in the r, θ plane:

$$f_r^{(2)} = \frac{16(\gamma+1)}{9R}(-r^7 + 6r^5 - 9r^3 + 4r), \tag{10.15}$$

$$f_\theta^{(2)} = 0. \tag{10.16}$$

This shows that in addition to the main flow along the axis of the pipe there is a radial component which diverges from the centre outward and is symmetrical with respect to the axis of the pipe.

Substituting $g_0^{(2)}$ and $f_r^{(2)}$ respectively from (10.5) and (10.15) in (6.21), a simple reduction gives

$$\begin{aligned}\frac{1}{r}\frac{d}{dr}\left(r\frac{df_0^{(2)}}{dr}\right) = {} & \frac{\Pi^4}{288}(-2r^8 + 11r^6 - 18r^4 + 11r^2 - 2) \\ & - \frac{\Pi}{2}\left[\frac{2(\gamma+1)}{R} - \frac{1}{4}\right](1-r^2)^2 \\ & - \frac{\Pi^3}{3R^2}[(3\gamma+8) - 3(\gamma+2)r^2] \\ & + \frac{(\gamma+1)\Pi^4}{1152}(-3r^8 + 16r^6 - 27r^4 + 18r^2 - 4) + D_0^{(2)},\end{aligned} \tag{10.17}$$

Integrating we obtain

$$\begin{aligned}f_0^{(2)} = {} & \frac{\Pi^4}{288}\left(-\frac{r^{10}}{50} + \frac{11r^8}{64} - \frac{18r^6}{36} + \frac{11r^4}{16} - \frac{r^2}{2}\right) \\ & - \frac{\Pi^3}{3R^2}\left[\frac{(3\gamma+8)}{4}r^2 - \frac{3(\gamma+2)}{16}r^4\right] \\ & + \frac{(\gamma+1)\Pi^4}{1152}\left[-\frac{3r^{10}}{100} + \frac{r^8}{4} - \frac{3r^6}{4} + \frac{9r^4}{8} - r^2\right] \\ & + \left[\frac{8(\gamma+1)}{R} - 1\right]\left[\frac{r^2}{4} - \frac{r^4}{8} + \frac{r^6}{36}\right] + \frac{D_0^{(2)}}{4}r^2 + D_1^{(2)},\end{aligned} \tag{10.18}$$

where the constants of integration are chosen to satisfy (6.22) and (6.23). Hence two algebraic equations result, which have the solution:

$$D_0^{(2)} = \frac{11\Pi^4}{30 \times 288} - \frac{7(\gamma+1)\Pi^4}{5 \times 1152} + \left[\frac{4(\gamma+1)l}{R} - \frac{1}{2}\right] + \frac{2(\gamma+3)\Pi^3}{3R^2}. \tag{10.19}$$

$$D_1^{(2)} = \frac{331\Pi^4}{480 \times 288} + \frac{151(\gamma+1)\Pi^4}{200 \times 1152} + \frac{1}{9}\left[\frac{2(\gamma+1)l}{R} - \frac{1}{4}\right] + \frac{(\gamma+2)\Pi^3}{48R^2}. \tag{10.20}$$

With the constants so determined we have for $f_0^{(2)}$:

$$
\begin{aligned}
f_0^{(2)} = {} & \frac{8}{2700}[-96r^{10} + 825r^8 - 2400r^6 + 3300r^4 - 19600r^2 + 331] \\
& + \frac{4(\gamma+1)}{45}[-12r^{10} + 100r^8 - 300r^6 + 450r^4 - 540r^2 + 302] \\
& + \left[\frac{(\gamma+1)l}{9R} - \frac{1}{72}\right][-2r^6 + 9r^4 - 9r^2 + 2] \\
& + \frac{32(\gamma+2)}{3R^2}[3r^4 - 4r^2 + 1],
\end{aligned} \tag{10.21}
$$

which, together with $g_0^{(2)}$, will give $V_0^{(2)}$ by (6.3).

The corresponding pressure $p^{(2)}$ is given by

$$
\begin{aligned}
p^{(2)} = {} & 2P_0(l - x_0)(1 - r^2) + P_1(l - x_0) \\
& + P_2(l - x_0)^2 + P_3(l - x_0)^3 + \frac{128}{27R^2}[2r^6 - 9r^4 + \tfrac{5}{2}],
\end{aligned} \tag{10.22}
$$

where

$$
P_0 = -\frac{512(\gamma+2)}{3R^2}, \qquad P_1 = \frac{2}{9}\left[\frac{344\gamma + 71}{270} + \frac{252(\gamma+1)l}{R} + \frac{1024(\gamma+3)}{R^2}\right]
$$

$$
P_2 = -\frac{256(\gamma+3)}{R^2}, \qquad P_3 = \frac{2048(\gamma+2)}{3R^3}.
$$

11. Drag and flux

As the velocity and pressure have already been found approximately, we can now calculate the corresponding drag and flux. This calculation is reduced to the calculation of their respective coefficients (7.3), (7.8). For the rth term of the drag coefficient, we find

$$
C_D^{(r)} = -\frac{4}{RS}\int_0^l \left\{\sum_{s=0}^{s=R} \frac{d}{dr}\left(\sigma^{(s)} V^{(r-s)}\right)\right\}_{r=1} dx_0. \tag{11.1}
$$

Substituting $\sigma^{(s)}$ and $V_0^{(r)}$, given in the previous sections, in the above equation, we obtain the following:

$$
C_D^{(0)} = \frac{16}{R}, \tag{11.2}
$$

$$
C_D^{(1)} = -\frac{128l}{R^2} + \frac{32}{R}, \tag{11.3}
$$

$$
\begin{aligned}
C_D^{(2)} = {} & \frac{169200\gamma + 675774}{1215R} - \frac{24(\gamma+1)l}{9R} + \frac{2048(\gamma+2)l^2}{R^2} \\
& + \frac{256(9\gamma+5)l}{3R^2} - \frac{512(\gamma+1)}{3R^3}.
\end{aligned} \tag{11.4}
$$

The complete expression for C_D will be of the form:

$$C_D = \frac{16}{R} - \left(\frac{128l}{R^2} - \frac{32}{R}\right)M^2 + \left[\frac{169200\gamma + 675774}{1215R} - \frac{24(\gamma+1)l}{9R} + \frac{2048(\gamma+2)l^2}{R^2} + \frac{256(9\gamma+5)l}{3R^2} - \frac{512(\gamma+1)}{3R^3}\right]M^4 + \cdots. \tag{11.5}$$

The first term gives the usual frictional drag for an incompressible fluid, while the other terms represent the contribution of compressibility. That this effect is positive or negative depends on the relative magnitude of l and R.

For the flux coefficients (7.8), we have

$$C_Q^{(r)} = 2lR\int_0^1 (p^{(r)})_{x_0=0} r\,dr, \tag{11.6}$$

for the general term of the C_Q series; the first three terms have the following values:

$$C_Q^{(0)} = 16l^2, \tag{11.7}$$

$$C_Q^{(1)} = -128l^3 + 32l^2, \tag{11.8}$$

$$C_Q^{(2)} = \frac{1344\gamma + 71}{135}l^2 + \frac{248\gamma - 264}{R}l^3 + \frac{5632\gamma + 17408}{3R^2}l^2 + \frac{2048(\gamma+2)}{3R^2}l^4. \tag{11.9}$$

The complete expression for C_Q is

$$C_Q = 16l^2 - (128l^3 - 32l^2)M^2 + \left[\frac{1344\gamma + 71}{135}l^2 + \frac{248\gamma - 264}{R}l^3 + \frac{5632\gamma + 17408}{3R^2}l^2 + \frac{2048(\gamma+2)}{3R^2}l^4\right]M^4 + \cdots. \tag{11.10}$$

The flux coefficient is reduced by compressibility. That is, for a given pressure head, the flux is increased by compressibility.

12. Summary

The problem of the subsonic flow through a pipe of a general section has been treated by the method of series development in ascending even powers of Mach number. Zero approximation is Hagen-Poiseuille motion. The solutions of the equations for the first and second approximations are all reduced to the problem of Dirichlet and the biharmonic problem with specified boundary conditions. The method has been applied to the case of a pipe of circular section for which explicit solutions have been obtained.

The paper was written while the author held a Sino-British Indemnity Fund Scholarship in the Department of Applied Mathematics at the University of Toronto. He wishes to take this opportunity to express his gratitude to Prof. J. L. Synge for suggesting the problem and for his constant guidance in the course of the work.

ON THE CONSTRUCTION OF CURVES OF CONSTANT WIDTH

By Brewster H. Gere and David Zupnik

1. Definition and general properties. In a paper by E. Barbier,[1] a curve of constant width is defined as a closed curve with a constant distance between parallel tangents. A method of construction, accredited by Barbier to Puiseux, was given which will produce examples of such curves other than circles. By taking half of an ellipse, where the line of division is the major axis, and marking off on each normal a distance equal to the major axis, such a curve can sometimes be obtained.

The following properties of a curve of constant width were stated.

Property 1. The curve is composed of two arcs with a common evolute.

Property 2. The common evolute of the two arcs must be a curve with one and only one tangent in each direction.[2]

Property 3. The envelope of the family of lines each of which lies midway between parallel tangents to the curve will be a curve with one and only one tangent in each direction.

Property 4. All curves of constant width with the same distance between parallel tangents will have the same length.

The reasoning used by Barbier in the proof of Property 4 was based upon probability considerations. Another derivation can be based upon the idea of average radius of curvature, that is, the ratio of arc length to difference of inclination between the end points of an arc. Suppose P and P' are opposite points of a curve of constant width. That is, P and P' lie on the same normal. Let the line PP' be of length $2a$, the width of the curve. If R is the radius of curvature at P and R' is the radius of curvature at P',

$$R + R' = 2a,$$

for there is a common point on the evolute for opposite points of the curve. Pairing opposite points, the average radius of curvature for the complete closed curve is seen to be a. The total difference of inclination is 2π. The length of a curve of constant width $2a$ is therefore $2\pi a$.

The problem considered in this paper is to find in general the type of configuration which may be used to construct a curve of constant width. Exact conditions upon the parts of such a configuration are presented and a general method of construction is given.

2. Evolute star. An evolute star is a configuration of n points or vertices $\{V_i\}$ and n curvilinear sides $\{\mathfrak{S}_i\}$ joining pairs of vertices. A curvilinear side

[1] Note sur le problème de l'aiguille, by E. Barbier, Journal de mathématiques pures et appliquées, 2nd series, Volume 5 (1860), pp. 273–286.

[2] As stated, this does not include the circle or a curve composed of arcs of circles. Since a circle is a curve of constant width, it will be shown later how such cases can be classed as degenerate.

may be a continuous convex arc, a straight line, or a continuous combination of convex arcs and straight lines. It must be of finite length. The tangent exists at every point of such an arc $\mathfrak{S}_i$ and must not cross the arc, although it may coincide with part of the arc if there is a straight portion. Moreover, while moving along the arc, the tangent must rotate always in the same direction through an angle less than π. The half tangent which extends in the direction of motion along the arc will be called the positive half tangent. A vertex is the point of intersection of two such arcs where the difference in inclination for the positive half tangents of the two arcs is π at that point. The inclination of any half tangent is an angle between 0 and 2π.

Let us choose a vertex V_1 and the tangent to side $\mathfrak{S}_1$ through V_1 as the base line for measuring angles of inclination. Consider the positive half tangent to $\mathfrak{S}_1$ at V_1 to have an inclination of π, the negative half tangent an inclination of zero. Let the inclination of $\mathfrak{S}_1$ increase from π to $\pi + \theta_2$ in going from V_1 to V_2 along $\mathfrak{S}_1$. At the point V_2 the inclination of the positive half tangent changes from $\pi + \theta_2$ to θ_2. In going from V_2 to V_3 along $\mathfrak{S}_2$ the inclination will increase from θ_2 to θ_3. At V_3 the inclination changes to $\pi + \theta_3$ and increases along $\mathfrak{S}_3$ to $\pi + \theta_4$, changing to θ_4 at V_4 and so forth. The side $\mathfrak{S}_i$ joins V_i and V_{i+1} except $\mathfrak{S}_n$, which joins V_n and V_1. The inclination of $\mathfrak{S}_n$ increases from $\pi + \theta_n$ to 2π. If $\mathfrak{S}_i$ is a straight line, $\theta_i = \theta_{i+1}$, but the word "increase" should be interpreted to include this case. It can be seen that

$$\pi \geqslant \theta_n \geqslant \theta_{n-1} \geqslant \cdots \geqslant \theta_2 \geqslant \theta_1 = 0.$$

Since the star is closed and the direction of the positive half tangent changes at each vertex, including V_1, n is odd.

3. Method of construction. The analysis of the construction of a curve of constant width will be carried through for the seven sided evolute star shown in Figure 1. The construction of curves by means of the general n-sided star can be done in precisely the same manner.

A taut string of convenient length k_1 is attached at V_1. On the base line of inclination π, or the positive half tangent to $\mathfrak{S}_1$ at V_1, the point P_1 is obtained where $V_1P_1 = k_1$. Winding the string around $\mathfrak{S}_1$, the end will trace an arc P_1Q_2. Recalling that the positive half tangent at V_i extends in the direction along $\mathfrak{S}_i$ toward V_{i+1}, in general P_i will be on the positive half tangent to the star at V_i and Q_i will be on the negative half tangent. Let

$$V_iP_i = k_i, \qquad V_iQ_i = h_i, \qquad \text{the length of } \mathfrak{S}_i = \lambda_i.$$

With this notation

$$k_1 = \lambda_1 + h_2.$$

A string of length $\lambda_2 + h_2$ is then to be fastened at V_2, unwound from $\mathfrak{S}_2$ and wound upon $\mathfrak{S}_3$ describing the arc $Q_2P_3Q_4$. A string of length $h_4 + \lambda_4$ is then fastened at V_5, unwound from $\mathfrak{S}_4$ and wound upon $\mathfrak{S}_5$ describing the arc $Q_4P_5Q_6$. A string of length $h_6 + \lambda_6$ is fastened at V_7, unwound from $\mathfrak{S}_6$ and wound upon $\mathfrak{S}_7$, describing the arc $Q_6P_7Q_1$.

For the half of the curve thus far constructed, the following relationships are seen to hold.

(A)
$$\text{Length of string at } V_i = k_i \qquad (i \text{ odd})$$
$$k_i = h_{i-1} + \lambda_{i-1} \qquad (i = 3, 5, \cdots, n)$$
$$h_j = k_{j-1} - \lambda_{j-1} \qquad (j = 2, 4, \cdots, n-1)$$
$$h_1 = k_n - \lambda_n$$

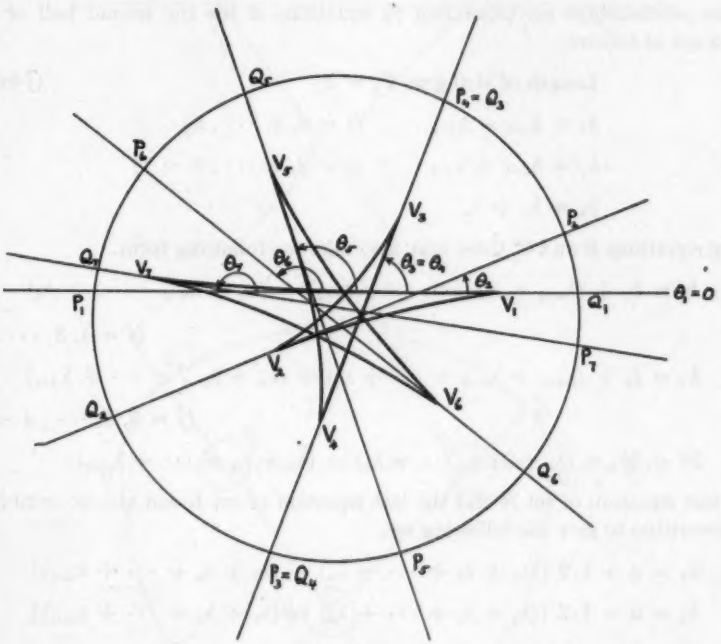

FIG. 1

Letting
$$2a = h_1 + k_1 = P_1Q_1,$$
these formulas may also take the following form.

(B)
$$k_i = k_1 + (\lambda_2 + \lambda_4 + \cdots + \lambda_{i-1}) - (\lambda_1 + \lambda_3 + \cdots + \lambda_{i-2}) \qquad (i = 1, 3, \cdots, n)$$
$$h_j = k_1 + (\lambda_2 + \lambda_4 + \cdots + \lambda_{j-2}) - (\lambda_1 + \lambda_3 + \cdots + \lambda_{j-1}) \qquad (j = 2, 4, \cdots, n-1)$$
$$h_1 = k_1 + (\lambda_2 + \lambda_4 + \cdots + \lambda_{n-1}) - (\lambda_1 + \lambda_3 + \cdots + \lambda_n)$$
$$2a - 2k_1 = (\lambda_2 + \lambda_4 + \cdots + \lambda_{n-1}) - (\lambda_1 + \lambda_3 + \cdots + \lambda_n)$$

In order to complete the curve, it will be necessary to fasten a string at each even numbered vertex. The vertex V_1 must be used again with a string of length k_1 since the arc previously described was entirely on one side of the positive half tangent at V_1. The strings are to be unwound from and wound upon the evolute arcs as before. The strings must be so chosen that

$$\text{(C)} \qquad \begin{aligned} h_i + k_i &= 2a \qquad (i = 1, 3, \cdots, n) \\ h_j + k_j &= 2a \qquad (j = 2, 4, \cdots, n-1) \end{aligned}$$

The relationships corresponding to equations A for the second half of the curve are as follows:

$$\text{(D)} \qquad \begin{aligned} &\text{Length of string at } V_j = k_j \qquad (j \text{ even}) \\ &h_i = k_{i-1} - \lambda_{i-1} \qquad (i = 3, 5, \cdots, n) \\ &k_j = h_{j-1} + \lambda_{j-1} \qquad (j = 2, 4, \cdots, n-1) \\ &k_1 = h_n + \lambda_n \end{aligned}$$

Using equations B and C, these may also take the following form.

$$\text{(E)} \qquad \begin{aligned} h_i &= k_1 + (\lambda_{n-1} + \lambda_{n-3} + \cdots + \lambda_{i+1}) - (\lambda_n + \lambda_{n-2} + \cdots + \lambda_i) \\ &\qquad\qquad (i = 1, 3, \cdots, n) \\ k_j &= k_1 + (\lambda_{n-1} + \lambda_{n-3} + \cdots + \lambda_j) - (\lambda_n + \lambda_{n-2} + \cdots + \lambda_{j+1}) \\ &\qquad\qquad (j = 2, 4, \cdots, n-1) \\ 2a - 2h_1 &= (\lambda_1 + \lambda_3 + \cdots + \lambda_n) - (\lambda_2 + \lambda_4 + \cdots + \lambda_{n-1}) \end{aligned}$$

The last equation of set B and the last equation of set E can also be combined and rewritten to give the following set.

$$\text{(F)} \qquad \begin{aligned} k_1 &= a + 1/2\,\{(\lambda_1 + \lambda_3 + \cdots + \lambda_n) - (\lambda_2 + \lambda_4 + \cdots + \lambda_{n-1})\} \\ h_1 &= a - 1/2\,\{(\lambda_1 + \lambda_3 + \cdots + \lambda_n) - (\lambda_2 + \lambda_4 + \cdots + \lambda_{n-1})\} \\ k_1 - h_1 &= (\lambda_1 + \lambda_3 + \cdots + \lambda_n) - (\lambda_2 + \lambda_4 + \cdots + \lambda_{n-1}) \end{aligned}$$

Since the only requirements to be satisfied in choosing the original length of string k_1 are that all the h's must be non-negative, and since the vertex chosen to be one in the numbering of vertices was any vertex, the following theorems have been proved.

THEOREM 1. Given an evolute star of n vertices, n an odd number, and n sides which, numbered in consecutive order, have lengths $\lambda_1, \lambda_2, \cdots, \lambda_n$, the necessary and sufficient condition that a curve of constant width $2a$ can be constructed is that $2a \geqslant \{(\lambda_1 + \lambda_3 + \cdots + \lambda_n) - (\lambda_2 + \lambda_4 + \cdots + \lambda_{n-1})\}$ for all possible choices of initial sides.

THEOREM 2. A given evolute star of n sides, n an odd number, can be used to construct one and only one family of parallel curves of constant width.

The proof of Theorem 2 is identical with the derivation of the formula for $k_1 - h_1$ since this difference is independent of the value of a.

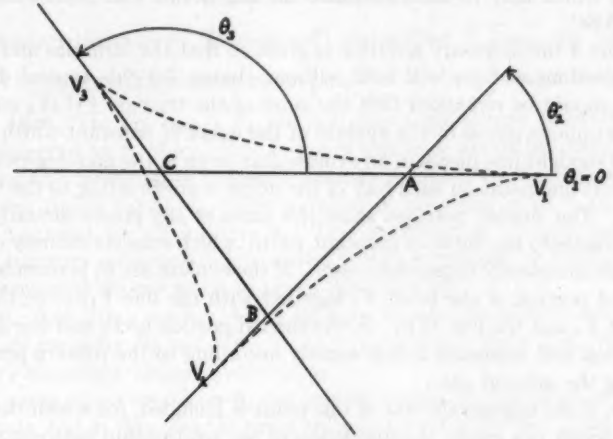

FIG. 2

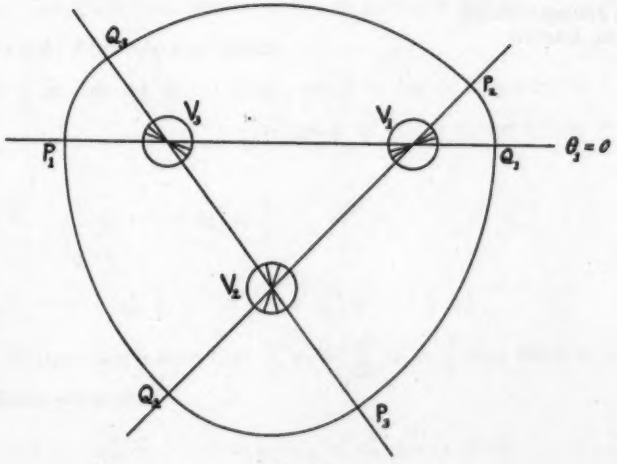

FIG. 3

4. Degenerate evolute star. An n sided configuration is a degenerate evolute star if, while otherwise fulfilling the conditions for an evolute star, any pair or pairs of consecutive sides fail to have a common tangent line.

It will be shown that by considering a part of a point as a portion of an arc, a degenerate star can be used in the same way as a non-degenerate star for constructing a curve of constant width. This can be seen most readily with a star

consisting of three points and the connecting lines. Further elaboration will be unnecessary. In Figure 2 the dotted lines can be considered as a non-degenerate star which may be deformed into the degenerate star represented by the triangle ABC.

In Figure 3 the necessary notation is given so that the formulas and theorems of the preceding sections will hold without change for this typical degenerate case. It should be remarked that the sides of the triangle $V_1V_2V_3$ merely join the discontinuous pieces of the evolute of the curve of constant width. This is true of all straight line pieces in an evolute star, even in the non-degenerate stars. There is but one point on each half of the curve corresponding to the whole line segment. The shaded portions show the parts of the points actually covered by the normals to the curve of constant width, which consists entirely of circular arcs in this completely degenerate case. If the evolute arc $\mathfrak{E}_1$ is considered to be the shaded portion of the point V_1 together with the line V_1V_2, $\mathfrak{E}_2$ the shaded portion of V_2 and the line V_2V_3, $\mathfrak{E}_3$ the shaded portion of V_3 and the line V_3V_1, the notation and reasoning follow exactly according to the pattern presented in discussing the general case.

Finally, if the degenerate star of one point is included, for which the curve of constant width is a circle, the discussion of the relationship between a curve of constant width and its evolute star is complete.

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ON THE CONVERGENCE OF TRIGONOMETRIC SERIES

By P. Erdös

A well known theorem of Kolmogoroff[1] states that if $\Sigma(a_{n_k}\cos n_k x + b_{n_k}\sin n_k x)$ is a lacunary trigonometric series $\left(\text{i.e.}\ \frac{n_{k+1}}{n_k} > c > 1\right)$ and if $\Sigma(a_{n_k}^2 + b_{n_k}^2)$ converges, then $\Sigma(a_{n_k}\cos n_k x + b_{n_k}\sin n_k x)$ converges almost everywhere. It has been observed by S. Sidon[2] that in many theorems about lacunary trigonometric series the condition of lacunarity can be replaced by the following condition (which Sidon called B_2): The number of solutions of $a = n_i + n_j$ is uniformly bounded for all integers a. This condition is trivially satisfied in the case of lacunary series and Sidon constructed B_2 sequences for which $n_k < ck$.[3] In the present note we are going to prove the following

Theorem. Let $\Sigma(a_{n_k}\cos n_k x + b_{n_k}\sin n_k x)$ be a trigonometric series where the n_k satisfy property B_2 and $\Sigma(a_{n_k}^2 + b_{n_k}^2)$ converges. Then $\Sigma(a_{n_k}\cos n_k x + b_{n_k}\sin n_k x)$ converges almost everywhere.

In most cases where condition B_2 can replace lacunarity the proof of this fact is very simple, but in this case the proof will not be quite trivial. First we need the following

Lemma. Let $P(x) = a_m\cos mx + \cdots + a_n\sin nx$ and let $\sum_m^n a_k^2 = \epsilon$. We can always split $P(x)$ into two parts

$$P(x) = (a_m\cos mx + \cdots + a_{l-1}\cos(l-1)x + a_l'\cos lx) + (a_l''\cos lx + \cdots + a_n\cos nx),\quad a_l = a_l' + a_l'' \tag{1}$$

such that

$$a_m^2 + \cdots + a_l'^2 \leqslant \frac{\epsilon}{2},\qquad a_l''^2 + \cdots + a_n^2 \leqslant \frac{\epsilon}{2} \tag{2}$$

and

$$a_m^2 + \cdots + a_l'^2 = a_l''^2 + \cdots + a_n^2\,. \tag{3}$$

Proof. If there is a q such that $\sum_m^q a_k^2 = \sum_{q+1}^n a_k^2 = \frac{\epsilon}{2}$ then there is nothing to prove. Hence suppose

$$a_m^2 + \cdots + a_{l-1}^2 < \frac{\epsilon}{2} < a_m^2 + \cdots + a_l^2$$

Let

$$\delta = \frac{\epsilon}{2} - (a_m^2 + \cdots + a_{l-1}^2),\qquad 0 < \delta < a_l^2.$$

[1] Fund. Math. 5. (1924) p. 96–97.
[2] Acta Szeged VII. (1934–35) p. 85–94.
[3] Sidon, ibid.

We have

$$a_{l+1}^2 + \cdots + a_n^2 = \epsilon - a_l^2 - \left(\frac{\epsilon}{2} - \delta\right) = \frac{\epsilon}{2} + \delta - a_l^2$$

Take

$$a_l' = \frac{\delta}{a_l}, \qquad a_l'' = a_l - \frac{\delta}{a_l}.$$

(1) is clearly satisfied. Now

$$a_m^2 + \cdots + a_l'^2 = \frac{\epsilon}{2} - \delta + \left(\frac{\delta}{a_l}\right)^2,$$

$$a_l''^2 + \cdots + a_n^2 = \left(a_l - \frac{\delta}{a_l}\right)^2 + \frac{\epsilon}{2} + \delta - a_l^2 = \frac{\epsilon}{2} - \delta + \left(\frac{\delta}{a_l}\right)^2.$$

Thus (3) is satisfied. (2) is also satisfied, since

$$\frac{\epsilon}{2} - \delta + \left(\frac{\delta}{a_l}\right)^2 < \frac{\epsilon}{2}$$

is an immediate consequence of $0 < \delta < a_l^2$.

COROLLARY. By repeating this process it is clear that we can divide $P(x)$ into 2^r consecutive parts (with splitting of terms) such that the sum Σa_k^2 for each part is inferior to $\frac{\epsilon}{2^r}$, and that all the sums are equal. Also clearly for r sufficiently large $\frac{\epsilon}{2^r} < \min(a_k^2)$ ($\min(a_k^2)$ meaning the minimum of those a_k^2 which are $\neq 0$). Thus each block of terms will ultimately contain no more than two terms $\neq 0$.

PROOF OF THE THEOREM. Without loss of generality we can assume that our series is a pure cosine series $\Sigma a_{n_k} \cos n_k x$. Put

$$\sum_{n_k=2^m}^{2^{m+1}-1} a_{n_k} \cos n_k x = P_m(x) = P_m, \qquad \sum_{2^m}^{2^{m+1}-1} a_{n_k}^2 = \epsilon_m$$

we have as a consequence of property B_2[3]

$$\int_0^{2\pi} P_m^4\, dx < c\left(\int_0^{2\pi} P_m^2\, dx\right)^2. \tag{4}$$

We know that $\sum_m P_m$ converges almost everywhere.[4] Now let us split $P_m(x)$ into 2^v parts as indicated in the lemma and the corollary, this operation being successively performed for $s = 1, 2, \cdots, v$. (v sufficiently large.)

Let $P_{m,s}$ be one of the 2^v parts obtained in the s-th operation. Let $E_{m,s}$ be the set of x for which

$$P_{m,s} > \frac{\epsilon_m^{1/4}}{2^{v/4}} \sqrt{s} \tag{5}$$

[4] Kolmogoroff, ibid.

We have by (4)

$$\int_0^{2\pi} P_{m,s}^4\,dx < c\,\frac{\epsilon_m^2}{2^{2s}}$$

and thus

$$|E_{m,s}| < c\,\frac{\epsilon_m}{2^s}\,\frac{!1}{\delta^2}, \qquad \text{where } |E| \text{ stands for the measure of } E.$$

Thus the sum of the 2^s, $|E_{m,s}|$ is less than $c\frac{\epsilon_m}{\delta^2}$ and the sum of all the $|E_{m,s}|$ ($s = 1, 2, \cdots v$) is less than $B\epsilon_m$, B being a constant.

It follows that the x for which (5) holds for infinitely many m is of measure 0.

Now if x does not belong to this set and if $2^m \leqslant q < 2^{m+1}$ it is easy to see that (by writing q in the binary scale)

$$\sum_{n_k=2^m}^{q} a_{n_k}\cos n_k x \leq \frac{\epsilon_m^{1/4}\sqrt{1}}{2^{1/4}} + \frac{\epsilon_m^{1/4}\sqrt{2}}{2^{2/4}} + \cdots \frac{\epsilon_m^{1/4}\sqrt{v}}{2^{v/4}}\,\frac{!1}{4}$$

$$+ \max_{2^m \leq n_k < 2^{m+1}} |a_{n_k}| < \delta(\delta \to 0)$$

which completes the proof of our theorem.

It is easy to see that instead of

$$\int_0^{2\pi} P_m^4\,dx < c\left(\int_0^{2\pi} P_m^2\,dx\right)^2$$

it would have been sufficient to assume that

$$\int_0^{2\pi} |P_m|^{2+\alpha}\,dx < c\left(\int_0^{2\pi} P_m^2\,dx\right)^{\frac{2+\alpha}{2}}$$

holds for any positive α.

It is clear from our proof that it suffices to show that our n_k are such that the number of solutions of $a = n_i + n_j$, $2^m < n_i, n_j < 2^{m+1}$ is uniformly bounded. Thus we can construct a sequence n_k which satisfies our condition and for which $n_k < k^{2.5}$

It might be of some number-theoretic interest to investigate whether our theorem remains true for $n_k = k^3$. Mordell[6] proved that $n_k = k^3$ does not satisfy condition B_2.

Another theorem of Kolmogoroff[7] states that if $\Sigma(a_n \cos nx + b_n \sin nx)$ is a trigonometric series such that $\Sigma(a_n^2 + b_n^2)$ converges, and if we put $\zeta_n = \sum_{k=0}^{n}(a_k \cos kx + b \sin kx)$. Then if $\frac{n_{k+1}}{n_k} > c > 1$, S_{n_k} converges almost everywhere. This result we can not prove if the n_k satisfy condition B_2.

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[5] P. Erdös and P. Turán, London Math. Soc. Journal 16, (1941) p. 212–215.

[6] Oral communication. See also K. Mahler, London Math. Soc. Proc. 39, (1935) p. 431–436.

[7] Kolmogoroff, ibid.

ON THE EXISTENCE OF PERIODIC SOLUTIONS FOR SECOND ORDER DIFFERENTIAL EQUATIONS WITH A FORCING TERM

By Norman Levinson

1. We shall consider the equation

$$\frac{d^2x}{dt^2} + f\left(x, \frac{dx}{dt}\right)\frac{dx}{dt} + g(x) = e(t), \tag{1.0}$$

or, as is more convenient for our purposes, the equvalent system

$$\frac{dx}{dt} = v, \qquad \frac{dv}{dt} = -f(x, v)v - g(x) + e(t). \tag{1.1}$$

Here $e(t)$ is a periodic function of t of period L. We shall show, subject to certain conditions on $f(x, v)$ and $g(x)$, that this system has at least one solution of period L.[1] Our result is sufficiently comprehensive to cover all cases of practical interest except for some cases of motion opposed by a Coulomb type of friction.

The following are the conditions under which we shall prove that (1.1) has a solution of period L: *$e(t)$ is continuous and $g(x)$ and $f(x, v)$ possess first order derivatives. $e(t)$ has period L. There exist an a and an m and M, all positive such that*

$$f(x, v) > m > 0 \quad \textit{when both } |x| > a \textit{ and } |v| > a, \tag{1.2}$$

and

$$f(x, v) > -M \quad \textit{otherwise.} \tag{1.3}$$

Also $xg(x) > 0$ for $|x| > a$ and

$$\lim_{|x|\to\infty} |g(x)| = \infty. \tag{1.4}$$

Moreover if

$$G(x) = \int_0^x g(x)\,dx, \tag{1.5}$$

then

$$\lim_{x\to\infty} \frac{g(x)}{G(x)} = 0. \tag{1.6}$$

[1] See S. Lefschetz, Existence of Periodic Solutions for Certain Differential Equations, Proc. Nat. Acad. of Science. Vol. 29, (1943), p. 29. Lefschetz gives a simple proof that the equation (a), $\ddot{x} + f(x)\dot{x} + g(x) = e(t)$, has at least one solution of the same period as $e(t)$. Lefschetz refers in a footnote to an unpublished more general result of Chevalley. Chevalley has informed me that his conditions on (a) include $\liminf f(x) > 0$ and $\liminf g(x)/x > 0$ $|x| \to \infty$. Thus Chevalley's results are of sufficiently general character to cover the van der Pol equation with a forcing term.

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These conditions can be relaxed considerably without affecting the following theorems but we shall not go into this here.

THEOREM I. *Subject to the conditions given above the equation* (1.0) *has at least one solution with the same period as the forcing term,* $e(t)$.

The periodic solution, the existence of which is guaranteed by Theorem I, need not be stable. In fact in many cases of interest this solution can be shown to be unstable. Nevertheless, even when unstable, it is often very useful to know of its existence.

We shall show that Theorem I follows easily from Theorem II. THEOREM II. *Subject to the conditions given above, there exist simple closed curves in the* (x, v) *plane such that a solution of* (1.1), $[x(t), v(t)]$, *can intersect any one of these curves only by crossing from the domain exterior to the curve into the domain interior to the curve. Moreover through any point in the* (x, v) *plane sufficiently remote from the origin, there passes a curve with this property.*

In many respects Theorem II is of more interest than Theorem I. Here however we shall use it merely as a step in the proof of Theorem I.

We shall now show that Theorem I is an almost immediate consequence of Theorem II. Let C be a closed curve with the properties described in Theorem II. Let P_0 be a point inside or on C. Let us consider the solution $(x_0(t), v_0(t))$ of (1.1) such that the point $(x_0(0), v_0(0))$ is the point P_0. Clearly $(x_0(t), v_0(t))$ lies inside of C for all $t > 0$. In particular $(x_0(L), v_0(L))$ lies inside of C. We denote $(x_0(L), v_0(L))$ by P_1 and define a transformation T as given by $TP_0 = P_1$. Clearly T is continuous and transforms the domain bounded by C into its interior. Thus according to the Brouwer fixed-point theorem, there exists at least one point P such that $TP = P$. This is equivalent to the statement that the solution of (1.1), $(x(t), v(t))$, which at $t = 0$ is at P, has period L. This proves Theorem I.

2. We turn now to the proof of Theorem II. Here we construct a curve, C, in the (x, v) plane such that any solution of (1.1), for which $(x(t_0), v(t_0))$ is a point on C, must lie in the domain interior to C for $t > t_0$.

We made use of the "energy" associated with the "motion" described in (1.1); that is we introduce

$$u(x, v) = \tfrac{1}{2}v^2 + G(x). \tag{2.0}$$

Clearly for large $|x|$, $G(x)$ is positive and increasing. In fact by (1.4) $G(x) \to \infty$ as $x \to \infty$. Thus for large values of u_0, the curves $u(x, v) = u_0$, where u_0 is a constant, are closed. It is also clear that if $u_1 > u_2$ then the curve $u(x, v) = u_2$ lies in the domain interior to $u(x, v) = u_1$.

By (1.1) we have

$$\frac{du}{dt} = -f(x, v)v^2 + e(t)v \tag{2.1}$$

along any solution of (1.1). This relationship will be very useful in proving Theorem II. We denote the max $|e(t)|$ by E. It is clear that our hypothesis remains valid if a is increased. Thus we can with no restriction assume

$$ma > 2E. \tag{2.2}$$

Proof of Theorem II. We now proceed to construct C, shown as $P_1P_2P_3 \ldots P_{11}P_{12}P_1$ in Fig. 1.

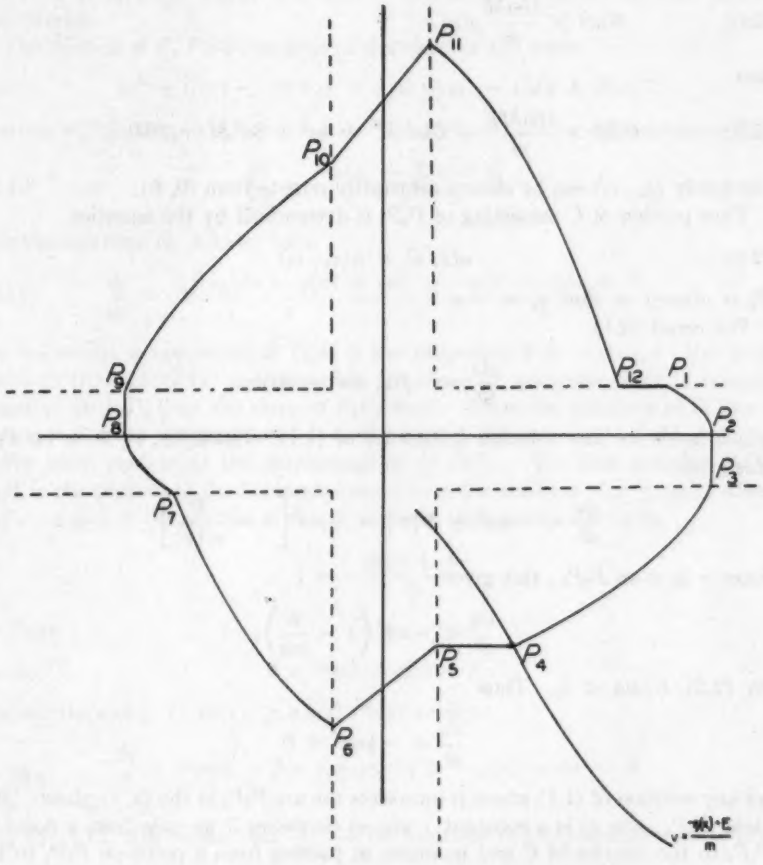

FIG. 1

We begin by locating P_4. This point lies in the curve

$$v = -\frac{g(x) + E}{m} \tag{2.3}$$

sufficiently far out so that if (x_4, v_4) denote its coordinates then for $x \geqq x_4$, $g(x) >> 0$ and $G(x) >> g(x)$. That this will be the case for large x_4 is clear from the hypothesis, in particular (1.4) and (1.6). To be more precise we require $x_4 > x_0$ where x_0 is chosen sufficiently large that for $x \geqq x_0$,

$$g(x) > 2(Ma + E), \qquad g(-x) < -2(Ma + E), \tag{2.4}$$

$$g(x) > \frac{2m}{M}(\max_{|x|\leq a} |g(x)| + E). \tag{2.5}$$

$$G(x) > \frac{16aM}{m} g(x), \tag{2.6}$$

and

$$G(x) > \frac{16aME}{m} + 64a^2M^2 + 4a^2 + 8a^3M + 2G(a). \tag{2.7}$$

Obviously (x_4, v_4) can be chosen arbitrarily remote from (0, 0).

That portion of C consisting of P_3P_4 is determined by the equation

$$u(x, v) = u(x_4, v_4). \tag{2.8}$$

P_3 is chosen so that $v_3 = -a$.

We recall (2.1)

$$\frac{du}{dt} = -f(x, v)v^2 + e(t)v$$

which holds for any solution $(x(t), v(t))$ of (1.1). Since $f(x, v) > m$ on P_3P_4 this gives

$$\frac{du}{dt} < -mv^2 + E|v| = -mv^2\left[1 - \frac{E}{m|v|}\right].$$

Since $v \geqq a$ on P_3P_4, this gives

$$\frac{du}{dt} < -mv^2\left(1 - \frac{E}{ma}\right).$$

By (2.2), $E/ma < \frac{1}{2}$. Thus

$$\frac{du}{dt} < -\tfrac{1}{2}mv^2 < 0$$

for any solution of (1.1) where it intersects the arc P_3P_4 in the (x, v) plane. But along P_3P_4, $u(x, v)$ is a constant. $u(x, v)$ decreases if we pass from a point on P_3P_4 to the interior of C and increases in passing from a point on P_3P_4 to the exterior of C. Therefore $\frac{du}{dt} < 0$ means that a solution of (1.1) can intersect the arc P_3P_4 only by passing from the domain exterior to C into the domain interior to C.[2]

The coordinates of P_3 are (x_3, v_3) and as already stated $v_3 = -a$. x_3 is determined from (2.8) on setting $v = -a$. P_2P_3 consists of the segment of the line

[2] Actually this is true for the open arc P_3P_4. Similarly in discussing other portions of C, P_nP_{n+1}, our proof will apply to the open arcs. That the solutions of (1.1) which start at any one of the isolated points, $P_1, P_2, \ldots, P_{12}$, must pass into the interior of C as t increases follows easily from the same inequalities as are developed to handle the several open arcs, $P_1P_2, P_2P_3, \ldots, P_{11}P_{12}, P_{12}P_1$.

$x = x_3$ for which $-a < v < 0$. Since $\frac{dx}{dt} = v < 0$ for $v < 0$ it follows that all solutions of (1.1) cut P_2P_3 from right to left, that is from the exterior of C to the interior.

The portion of C, P_1P_2, consists of that part of the curve

$$\tfrac{1}{2}v^2 + G(x) - (Ma + E)x = G(x_2) - (Ma + E)x_2 \tag{2.9}$$

starting at P_2 and for which $0 < v < a$. On P_1P_2 we have by differentiating (2.9)

$$\frac{dv}{dx} = \frac{-g(x) + Ma + E}{v}. \tag{2.10}$$

For the solutions of (1.1) we have

$$\frac{dv}{dx} = \frac{-f(x, v)v - g(x) + e(t)}{v} < \frac{-g(x) + Ma + E}{v}. \tag{2.11}$$

An immediate consequence of (2.4) is the inequality $0 > -g(x) + Ma + E$. Thus (2.10) and (2.11) indicate that the slope of solutions of (1.1) is more negative on P_1P_2 than the slope of P_1P_2 itself. Thus the solutions of (1.1) cut across P_1P_2 from the exterior of C to the interior.

We have completed the determination of P_1P_4. We turn next to P_4P_5. P_4P_5 is the portion of the horizontal line $v = v_4$ for which $a < x < x_4$. Along P_4P_5, in fact in the portion of the (x, v) plane lying below the curve

$$v = -\frac{g(x) + E}{m}$$

we have

$$0 < -mv - g(x) - E. \tag{2.12}$$

For solutions of (1.1) with $x \geqq a$ and $v < 0$ we have

$$\frac{dv}{dt} = -g(x) - f(x, v)v + e(t) > -g(x) - mv - E.$$

But by (2.12) this means that $\frac{dv}{dt} > 0$ for solutions of (1.1) as they cross P_4P_5. Thus, again, solutions of (1.1) can cut P_4P_5 only by passing from the exterior of C to the interior.

P_5P_6 consists of the portion of the line $v = 2M(x - a) + v_5$ for which $-a < x < a$. On this line

$$\frac{dv}{dx} = 2M. \tag{2.13}$$

For solutions of (1.1) as they cut P_5P_6 we have

$$\frac{dv}{dx} = \frac{-f(x, v)v - g(x) + e(t)}{v} < M + \frac{1}{|v_5|}\left(\max_{|x| \leq a} |g(x)| + E\right). \tag{2.14}$$

Since $v_4 = v_5$ we have by (2.3) and (2.5)

$$\frac{1}{|v_5|}\left(\max_{|x|\leq a}|g(x)| + E\right) < \frac{\frac{M}{2m}g(x_4)}{\frac{1}{m}(g(x_4)+E)} < \frac{M}{2}.$$

Thus (2.14) becomes

$$\frac{dv}{dx} < \frac{3}{2}M.$$

Comparing this with (2. 3) we see that solutions of (1.1) cutting P_5P_6 have a smaller slope than P_5P_6 itself. Thus solutions of (1.1) must cut across P_5P_6 from the exterior of C to the interior.

P_6P_7 is the portion of the curve $u(x, v) = u(x_6, v_6)$ with $v_7 = -a$. As in the case of P_3P_4, $\frac{du}{dt} < 0$ for solutions of (1.1) in this region. Thus solutions of (1.1) cut P_6P_7 from the exterior of C to the interior.

P_7P_8 much like P_1P_2, is the portion of the curve

$$\tfrac{1}{2}v^2 + G(x) + (Ma + E)x = \tfrac{1}{2}a^2 + G(x_7) + (Ma + E)x_7$$

for which $-a < v < 0$ and which starts at P_7. As in the case of P_1P_2 we make use of (2.4) here. If x_7 is not sufficiently large in magnitude for the second inequality of (2.4) to apply, we can increase $|x_7|$ by moving out P_4 so that (2.4) does apply, that is so that $|x_7| > x_0$.

P_8P_9 is the vertical line $x = x_8$ with $0 < v < a$.

P_9P_{10} is the portion of the curve $u(x, v) = u(x_9, v_9)$ with $x_{10} = -a$.

$P_{10}P_{11}$ is the portion of the straight line $v = 2M(x + a) + v_{10}$, with $x_{11} = a$.

$P_{11}P_{12}$ is the portion of the curve $u(x, v) = u(x_{11}, v_{11})$ with $v_{12} = a$.

It follows easily as in the cases already considered that these are all cut by solutions of (1.1) from the exterior of C to the interior. We now have to show that P_{12} lies to the left of P_1.

Once we show P_{12} lies to the left of P_1, it follows easily that solutions of (1.1) must cut the line $P_{12}P_1$ from the exterior of C to the interior. For on $P_{12}P_1$ we have for solutions of (1.1)

$$\frac{dv}{dt} = -f(x, v)v - g(x) + e(t) < -ma - g(x) + E.$$

Since $ma > 2E > E$ and since $g(x) > 0$ for $x > a$, we have $\frac{dv}{dt} < 0$. Thus $P_{12}P_1$ is cut by solutions of (1.1) from the exterior of C to the interior.

To show that P_{12} lies to the left of P_1 we consider the changes in u as we traverse C. We denote $u(x_n, v_n)$ by u_n. Setting $x = x_1$ and $v = v_1$ in (2.9) we get

$$G(x_2) - G(x_1) - \tfrac{1}{2}a^2 = (Ma + E)(x_2 - x_1) \tag{2.15}$$

or

$$u_2 - u_1 = (Ma + E)(x_2 - x_1). \tag{2.16}$$

From (2.15) we also have

$$\int_{x_1}^{x_2} g(x)\,dx = (Ma + E)(x_2 - x_1) + \tfrac{1}{2}a^2. \tag{2.17}$$

Now since $x_1 > x_4 > x_0$ we have from (2.4), $g(x) > 2(Ma + E)$ for $x \geqq x_1$. Thus (2.17) gives

$$2(Ma + E)(x_2 - x_1) < (Ma + E)(x_2 - x_1) + \tfrac{1}{2}a^2.$$

Or $(Ma + E)(x_2 - x_1) < \tfrac{1}{2}a^2$. Thus (2.16) becomes

$$u_2 - u_1 < \tfrac{1}{2}a^2. \tag{2.18}$$

Since P_2P_3 consists of the vertical line segment $x = x_2$, $-a < v < 0$, we have clearly

$$u_3 - u_2 = \tfrac{1}{2}a^2 \tag{2.19}$$

Since $u(x, v)$ is a constant along P_3P_4 we have

$$u_4 - u_3 = 0. \tag{2.20}$$

Since $v_4 = v_5$ we have

$$u_5 - u_4 = G(a) - G(x_4) \tag{2.21}$$

Using the definition of u we have

$$u_6 - u_5 = \tfrac{1}{2}v_6^2 + G(-a) - \tfrac{1}{2}v_5^2 - G(a).$$

From the definition of the line segment P_5P_6 we have $v_6 = -4aM + v_5$. Thus since $v_5 = v_4$, the above expression for $u_6 - u_5$ becomes

$$u_6 - u_5 = 8a^2M^2 - 4aMv_4 + G(-a) - G(a). \tag{2.22}$$

Clearly

$$u_7 - u_6 = 0 \tag{2.23}$$

Much as with $u_2 - u_1$ we have

$$u_8 - u_7 < \tfrac{1}{2}a^2 \tag{2.24}$$

and as with $u_3 - u_2$,

$$u_9 - u_8 = \tfrac{1}{2}a^2. \tag{2.25}$$

Clearly

$$u_{10} - u_9 = 0. \tag{2.26}$$

As in the case of (2.22) we get

$$u_{11} - u_{10} = 8a^2M^2 + 4aMv_{10} + G(a) - G(-a).$$

By using (2.23)–(2.26) we have

$$u_{10} - u_6 < a^2.$$

Since $x_6 = x_{10} = -a$ we have from this inequality

$$v_{10}^2 - v_6^2 < 2a^2,$$

or

$$v_{10} + v_6 < \frac{2a^2}{v_{10} - v_6},$$

or

$$v_{10} < |v_6| + \frac{2a^2}{v_{10} + |v_6|}.$$

Since $u_{10} > u_6$, $v_{10} > |v_6|$. Also clearly $|v_6| > |v_4|$ and we can take $|v_4| > 1$. Thus

$$v_{10} < |v_6| + a^2.$$

But as already seen $|v_6| = 4aM + |v_4|$. Thus $v_{10} < -v_4 + 4aM + a^2$, and using this in the equation for $u_{11} - u_{10}$ we get

$$u_{11} - u_{10} < -4aMv_4 + 24a^2M^2 + 4a^3M + G(a) - G(-a). \tag{2.27}$$

Finally

$$u_{12} - u_{11} = 0. \tag{2.28}$$

Adding (2.18)–(2.28) we get

$$u_{12} - u_1 < -G(x_4) - 8aMv_4 + 32a^2M^2 + 4a^3M + 2a^2 + G(a). \tag{2.29}$$

Since P_4 lies on the curve $mv = -(g(x) + E)$ we have

$$v_4 = -\frac{g(x_4) + E}{m}.$$

Thus (2.29) becomes

$$u_{12} - u_1 < -G(x_4) + \frac{8aM}{m} g(x_4) + \frac{8aME}{m} + 32a^2M^2 + 4a^3M + 2a^2 + G(a). \tag{2.30}$$

By (2.6)

$$-G(x_4) + \frac{8aM}{m} g(x_4) < -\tfrac{1}{2}G(x_4).$$

Thus (2.30) becomes

$$u_{12} - u_1 < -\tfrac{1}{2}\left[G(x_4) - \frac{16aME}{m} - 64a^2M^2 - 8a^3M - 4a^2 - 2G(a)\right].$$

By (2.7) it follows at once that

$$u_{12} - u_1 < 0.$$

In other words $G(x_{12}) < G(x_1)$. Since $G(x)$ is monotonically increasing for large x this implies $x_{12} < x_1$. Thus we have proved that P_{12} lies to the left of P_1 and thereby demonstrated the existence of the curves C of Theorem II.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

TABLE OF COEFFICIENTS IN NUMERICAL INTEGRATION FORMULAE

By A. N. Lowan and Herbert Salzer

The values of $B_n^{(n)}(1)/n!$ and $B_n^{(n)}/n!$, where $B_n^{(n)}(1)$ denotes the n^{th} Bernoulli polynomial of the n^{th} order for $x = 1$ and $B_n^{(n)}$ denotes the n^{th} Bernoulli number of the n^{th} order, were computed for $n = 1, 2, \cdots 20$. The quantities $B_n^{(n)}(1)/n!$ are required in the Laplace formula of numerical integration employing forward differences, as well as in the Gregory formula. The quantities $B_n^{(n)}/n!$ are used in the Laplace formula employing backward differences. (See Milne-Thomson, "Calculus of Finite Differences", pp. 181–184, 191–193.)

n	$B_n^{(n)}(1)/n!$	$B_n^{(n)}/n!$
1	$\frac{1}{2}$	$-\frac{1}{2}$
2	$-\frac{1}{12}$	$\frac{5}{12}$
3	$\frac{1}{24}$	$-\frac{3}{8}$
4	$-\frac{19}{720}$	$\frac{251}{720}$
5	$\frac{3}{160}$	$-\frac{95}{288}$
6	$-\frac{863}{60480}$	$\frac{19087}{60480}$
7	$\frac{275}{24192}$	$-\frac{5257}{17280}$
8	$-\frac{33953}{36288\ 00}$	$\frac{10700\ 17}{36288\ 00}$
9	$\frac{8183}{10368\ 00}$	$-\frac{25713}{89600}$
10	$-\frac{32504\ 33}{47900\ 1600}$	$\frac{26842\ 253}{95800\ 320}$
11	$\frac{4671}{78848\ 0}$	$-\frac{47772\ 23}{17418\ 240}$
12	$-\frac{13695\ 77909\ 3}{26153\ 48736\ 000}$	$\frac{70360\ 42543\ 57}{26153\ 48736\ 000}$
13	$\frac{22242\ 34463}{47551\ 79520\ 00}$	$-\frac{10636\ 47638\ 17}{40236\ 13440\ 00}$
14	$-\frac{13228\ 28401\ 27}{31384\ 18483\ 2000}$	$\frac{11663\ 09819\ 657}{44834\ 54976\ 000}$

n	$B_n^{(n)}(1)/n!$	$B_n^{(n)}/n!$
15	26396 51053 68976 23040 00	−25221 445 98402 304
16	−11195 67034 48001 32011 86852 86400 00	80929 89203 53324 9 32011 86852 86400 00
17	50188 465 15613 16556 8	−85455 47771 5379 34237 29254 40000
18	−23340 28946 34446 3 78601 44949 49376 000	12600 46723 60427 56559 51090 94217 17094 40000
19	30112 40351 85049 10928 54378 00448 000	−13115 46499 95723 6437 53779 93912 81152 0000
20	−12365 72232 34699 80029 48171 45976 18974 72000 00	81368 36498 46758 25997 87 33720 02183 33282 30400 000

National Bureau of Standards,
Mathematical Tables Project, New York City

TABLE OF $Ji_0(x) = \int_x^\infty \frac{J_0(t)}{t}\,dt$ AND RELATED FUNCTIONS[1]

By Arnold N. Lowan, G. Blanch, and M. Abramowitz

Requests for the investigation of the above integral[2] have come to the Mathematical Tables Project from various quarters; because of the possible importance of the integral in various physical applications, the authors are publishing here tabular values of the function over a considerable region, together with closely related functions and an asymptotic expansion of the integral.

I. Derivation of Series Expansion for $Ji_0(x)$

From the known expansion of $J_0(t)$, we have

$$\begin{aligned}\int_x^\infty (J_0(t)/t)\,dt &= \int_x^\infty \left[(1/t) + \sum_{n=1}^\infty (-1)^n t^{2n-1}/(n!)^2 2^{2n}\right] dt \\ &= K - \log_e x + \sum_{n=1}^\infty (-1)^{n-1}(x/2)^{2n}/2n(n!)^2\end{aligned} \tag{1}$$

where

$$K = \lim_{t\to\infty}\left[\log_e t + \sum_{n=1}^\infty (-1)^n (t/2)^{2n}/2n(n!)^2\right].$$

To determine K, make use of the expression

$$J_0(t) = (2/\pi)\int_0^{\pi/2} \cos(t\sin u)\,du.$$

Hence

$$\begin{aligned}Ji_0(x) &= (2/\pi)\int_x^\infty (1/t)\,dt \int_0^{\pi/2} \cos(t\sin u)\,du \\ &= (2/\pi)\int_0^{\pi/2} du \int_x^\infty [\cos(t\sin u)/t]\,dt.\end{aligned} \tag{2}$$

Now

$$Ci(x) = -\int_x^\infty (\cos t/t)\,dt = \gamma + \log_e x - x^2/2.2! + \cdots$$

[1] The results reported here were obtained in the course of the work done by the Mathematical Tables Project conducted by the Work Projects Administration for New York City under the sponsorship of the National Bureau of Standards, Dr. Lyman J. Briggs, Director.

[2] The notation $Ji_0(x)$ for the integral was introduced by V. G. Smith. See his paper in this issue.

Hence (2) becomes

$$\begin{aligned} Ji_0(x) &= -(2/\pi)\int_0^{\pi/2} Ci(x\sin u)\,du \\ &= (2/\pi)\int_0^{\pi/2} -[\gamma + \log_e x + \log_e \sin u + f(x\sin u)]\,du \end{aligned} \tag{3}$$

where $f(x \sin u)$ is a convergent power series in $x \sin u$ (beginning with $-x^2 \sin^2 u/2!2$).

It is apparent by inspection that (3) must coincide with (1). Since we are interested only in determining the constant term K, we shall confine our attention to the relevant first and third terms of the integral (3).

This gives

$$K = -(2/\pi)\int_0^{\pi/2} \gamma\,du + (2/\pi)\int_0^{\pi/2} -\log_e \sin u\,du.$$

It can be shown that

$$(2/\pi)\int_0^{\pi/2} \log_e \sin u\,du = -\log_e 2.$$

Hence

$$K = -\gamma + \log_e 2.$$

Therefore

$$\begin{aligned} Ji_0(x) &= -\gamma + \log_e 2 + \left[\sum_{n=1}^{\infty} (-1)^{n-1}(x/2)^{2n}/2n(n!)^2\right] - \log_e x \\ &= F(x) - \log_e (x/2) \end{aligned}$$

where

$$F(x) = -\gamma + \sum_{n=1}^{\infty} (-1)^{n-1}(x/2)^{2n}/2n(n!)^2.$$

In the above, $\gamma = .57721\ 56649\ 015$ is Euler's constant to 13 decimals.

II. Asymptotic Expansions

The following asymptotic expansion holds for $Ji_0(x)$:

$$Ji_0(x) \sim \sqrt{2/\pi x}\left\{\cos\left(x - \frac{\pi}{4}\right)P(x) + \sin\left(x - \frac{\pi}{4}\right)Q(x)\right\}$$

or sometimes more conveniently

$$Ji_0(x) \sim \sqrt{1/\pi x}\,\{(\sin x + \cos x)P(x) + (\sin x - \cos x)Q(x)\}$$

where

$$P(x) = \frac{1.625}{x^2} - \frac{14.53808\ 59375}{x^4} + \frac{360.66157\ 15027}{x^6} - \frac{17588.27309\ 892}{x^8} + \frac{1420335.136666}{x^{10}} - \frac{171507284.2854}{x^{12}} + \cdots$$

$$Q(x) = -\frac{1}{x} + \frac{4.1328125}{x^3} - \frac{65.53353\ 881836}{x^5} + \frac{2344.872716188}{x^7} - \frac{149506.3953828}{x^9} + \frac{14913628.95213}{x^{11}} - \cdots$$

In the above, the first two coefficients of $P(x)$ and $Q(x)$ are exact. Let us designate $P(x)$ and $Q(x)$ as follows:

$$P(x) = \sum_{r=1}^{\infty} A_r/x^{2r}; \qquad Q(x) = \sum_{r=0}^{\infty} B_r/x^{2r+1}$$

where

$$A_1 = \frac{13}{8}, \qquad A_2 = \frac{-14887}{2^{10}}, \qquad A_3 = \frac{94545267}{2^{18}}, \qquad A_4 = \frac{-590164513695}{2^{25}}$$

$$A_r = C_{r-1} - \frac{(4r-1)(4r-3)}{4} A_{r-1}, \qquad \text{and}$$

$$C_r = \frac{(-1)^r 1^2 3^2 \cdots (4r-1)^2 (48r^2 + 48r + 13)}{(2r+1)!\, 2^{6r+3}}$$

Similarly

$$B_0 = -1; \qquad B_1 = \frac{529}{2^7}; \qquad B_2 = -\frac{2147403}{2^{15}};$$

$$B_3 = \frac{9835109013}{2^{22}}; \qquad B_4 = \frac{-321062539355955}{2^{31}}$$

$$B_r = -a_r - \frac{(16r^2-1)}{4}(B_{r-1} + a_{r-1}) + \frac{(4r+1)}{2} C_{r-1} \qquad \text{and}$$

$$a_r = \frac{(-1)^r 1^2 3^2 \cdots (4r-1)^2}{(2r)!\, 2^{6r}}.$$

The reader should compare the above with the elegant asymptotic expression given by V. G. Smith in this issue. Smith's expansion and the one given here converge with the same rapidity; the former is convenient over the region where $J_0(x)$ and $J_1(x)$ are tabulated (up to $x = 25$)[3]; beyond that range, the expressions given here might be more convenient.

[3] British Association Adv. Sci., *Mathematical Tables*, Vol. VI, Cambridge Univ. Press (1937).

III. Description of Tabular Material

Notation for indicating range and interval between successive arguments will follow customary usage. Thus $x = 0(.1)3(1)22$ indicates a range of x from 0 to 3 at intervals of 0.1 and from 3 to 22 at intervals of unity.

TABLE I:

$Ji_0(x)$ to 10 decimal places and $F(x) = Ji_0(x) + \log_e \frac{1}{2}x$ to 12 decimal places for $x = 0(.1)3$ with even central differences of $F(x)$.

TABLE I

$$Ji_0(x) = \int_x^\infty \frac{J_0 t}{t}\, dt;\ F(x) = Ji_0(x) + \log_e (x/2)$$

x	$Ji_0(x)$	$F(x)$	$\delta^2 F(x)$	$\delta^4 F(x)$	$\delta^6 F(x)$
0.0	∞	−0.577215 664902	+2499 218896	−9 366328	+52012
.1	+2.41976 62181	−0.575966 055454	2494 535732	9 340322	51818
.2	1.73036 31827	−0.572221 910274	2480 512246	9 262498	51318
.3	1.33112 27320	−0.565997 252848	2457 226262	9 133356	50473
.4	1.05212 25433	−0.557315 369160	2424 806922	8 953741	49299
0.5	+0.84008 56826	−0.546208 678550	+2383 433841	−8 724827	+47784
.6	0.67125 42502	−0.532718 554099	2333 335933	8 448129	46002
.7	0.53292 70308	−0.516895 093715	2274 789896	8 125429	43874
.8	0.41749 38884	−0.498796 843435	2208 118430	7 758855	41492
.9	0.32001 72215	−0.478490 474725	2133 688109	7 350789	38845
1.0	+0.23709 67627	−0.456050 417906	+2051 906999	−6 903878	+35927
1.1	0.16627 85467	−0.431558 454088	1963 222011	6 421040	32815
1.2	0.10572 23555	−0.405103 268259	1868 115983	5 905387	29482
1.3	0.05400 29496	−0.376779 966447	1767 104568	5 360252	25957
1.4	+0.00998 53839	−0.346689 560067	1660 732901	4 789160	22318
1.5	−0.02725 63483	−0.314938 420786	+1549 572074	−4 195750	+18496
1.6	0.05849 41581	−0.281637 709431	1434 215497	3 583844	14616
1.7	0.08438 38531	−0.246902 782579	1315 275076	2 957322	10646
1.8	0.10549 20650	−0.210852 580651	1193 377333	2 320154	6617
1.9	0.12231 57070	−0.173609 001390	1069 159436	1 676369	+ 2589
2.0	−0.13529 62627	−0.135296 262693	+ 943 265170	−1 029995	− 1437
2.1	0.14483 04230	−0.096040 258826	816 340909	− 385058	5432
2.2	0.15127 80939	−0.055967 914050	689 031590	+ 254447	9358
2.3	0.15496 84801	−0.015206 537684	561 976718	884594	13171
2.4	0.15620 47414	+0.026116 815400	435 806440	1 501570	16918
2.5	−0.15526 75764	+0.067875 974924	+ 311 137732	+2 101628	−20461
2.6	0.15241 79923	+0.109946 272180	188 570652	2 681225	23873
2.7	0.14789 94524	+0.152205 140088	+ 68 684797	3 236949	27099
2.8	0.14193 95438	+0.194532 692793	− 47 964109	3 765574	30083
2.9	0.13475 12750	+0.236812 281389	160 847441	4 264116	32868
3.0	−0.12653 40856	+0.278931 022544	− 269 466657	+4 729790	−35397

TABLE II

$$Ji_0(x) = \int_x^\infty \frac{J_0 t}{t}\, dt$$

x	$Ji_0(x)$	δ^2	δ^{4*}	x	$Ji_0(x)$	δ^2	δ^{4*}
3.0	−0.12653 40856	+84 22623	+1 21371	7.0	+0.01143 05636	+5 43574	−23237
3.1	.11747 46339	66 77684	1 16593	7.1	.00717 77942	9 45362	25810
3.2	.10774 74138	50 49542	1 12782	7.2	+0.00301 95610	13 21405	28062
3.3	.09751 52395	35 34334	1 09699	7.3	−0.00100 65317	16 69449	30017
3.4	.08692 96318	21 28933	1 07125	7.4	.00486 56795	19 87543	31639
3.5	−0.07613 11308	+ 8 30732	+1 04913	7.5	−0.00852 60730	+22 74062	−32959
3.6	.06524 95566	− 3 62512	1 02904	7.6	.01195 90603	25 27690	33944
3.7	.05440 42336	14 52828	1 01010	7.7	.01513 92786	27 47440	34610
3.8	.04370 41934	24 42131	99122	7.8	.01804 47529	29 32643	34977
3.9	.03324 83663	33 32321	97194	7.9	.02065 69629	30 82934	35027
4.0	−0.02312 57713	−41 25341	+ 95138	8.0	−0.02296 08795	+31 98261	−34769
4.1	.01341 57104	48 23251	92955	8.1	.02494 49700	32 78875	34250
4.2	−0.00418 79746	54 28248	90551	8.2	.02660 11730	33 25299	33431
4.3	+0.00449 69364	59 42733	87973	8.3	.02792 48461	33 38344	32367
4.4	.01258 75741	63 69295	85140	8.4	.02891 46848	33 19072	31059
4.5	+0.02004 12823	−67 10763	+ 82092	8.5	−0.02957 26163	+32 68788	−29521
4.6	.02682 39142	69 70188	78803	8.6	.02990 36690	31 89025	27780
4.7	.03290 95273	71 50859	75276	8.7	.02991 58192	30 81520	25854
4.8	.03828 00545	72 56300	71528	8.8	.02961 98174	29 48195	23766
4.9	.04292 49517	72 90257	67569	8.9	.02902 89961	27 91134	21531
5.0	+0.04684 08232	−72 56687	+ 63405	9.0	−0.02815 90614	+26 12566	−19183
5.1	.05003 10260	71 59748	59069	9.1	.02702 78701	24 14836	16731
5.2	.05250 52540	70 03774	54565	9.2	.02565 51952	22 00389	14216
5.3	.05427 91046	67 93261	49941	9.3	.02406 24814	19 71737	11647
5.4	.05537 36291	65 32832	45189	9.4	.02227 25939	17 31444	9045
5.5	+0.05581 48704	−62 27229	+ 40373	9.5	−0.02030 95620	+14 82104	−6460
5.6	.05563 33888	58 81267	35480	9.6	.01819 83197	12 26303	3869
5.7	.05486 37805	54 99828	30583	9.7	.01596 44471	9 66622	−1341
5.8	.05354 41894	50 87808	25670	9.8	.01363 39123	7 05589	+1138
5.9	.05171 58175	46 50111	20796	9.9	.01123 28186	4 45677	3531
6.0	+0.04942 24345	−41 91606	+ 15977	10	−0.00878 71572	+ 1 89276	+5827
6.1	.04670 98909	37 17105	11253	11	+0.01292 57668		
6.2	.04362 56368	32 31328	6635	12	+0.01878 07595		
6.3	.04021 82499	27 38886	+ 2167	13	+0.00763 73034		
6.4	.03653 69744	22 44243	− 2140	14	−0.00766 44333		
				15	−0.01356 79463		
6.5	+0.03263 12746	−17 51700	− 6255	16	−0.00689 35468		
6.6	.02855 04048	12 65367	10150	17	+0.00452 43666		
6.7	.02434 29983	7 89137	13821	18	+0.01024 04488		
6.8	.02005 66781	− 3 26675	17234	19	+0.00629 95028		
6.9	.01573 76904	+ 1 18609	20378	20	−0.00249 01501		
				21	−0.00791 41326		
7.0	+0.01143 05636	+ 5 43574	− 23237	22	−0.00577 42961		

* These differences are modified. (See *Mathematical Tables*, Vol. I, page xi, Br. Assn. Adv. Sci. for a discussion of modified differences.) For purposes of interpolation, modified differences are to be used exactly like ordinary differences.

TABLE III

Reduced Derivatives of $F(x)$ Values of $A_n(x) = F^{(n)}(x)/n!$
$A_0(x) = F(x) = Ji_0(x) + \log_e \frac{1}{2}x$

n	$x = 10$	n	$x = 11$	n	$x = 12$
0	+1.600650 755192	0	+1.717673 859029	0	+1.810540 228772
1	+ .124593 576445	1	+ .106471 845492	1	+ . 79359 224100
2	− . 4056 041514	2	− . 12875 324748	2	− . 12616 930358
3	− . 3900 981217	3	− . 1569 963797	3	+ . 1621 911448
4	+ . 217 533113	4	+ . 824 565647	4	+ . 649 913740
5	+ . 188 423521	5	+ . 42 758907	5	− . 101 078353
6	− . 16 493896	6	− . 28 041071	6	− . 16 733757
7	− . 3 287360	7	+ . 107170	7	+ . 2 791648
8	+ . 348081	8	+ . 437064	8	+ . 192446
9	+ . 30595	9	− . 11124	9	− . 38530
10	− . 3777	10	− . 3964	10	− . 1185
11	− . 181	11	+ . 143	11	+ . 322
12	+ . 26	12	+ . 24	12	+ . 4
13	− . 1	13	− . 1	13	− . 2

n	$x = 13$	n	$x = 14$	n	$x = 15$
0	+1.879439 480296	0	+1.938245 715759	0	+2.001335 074289
1	+ . 61005 684433	1	+ . 59209 037421	1	+ . 67614 964855
2	− . 5050 912944	2	+ . 2648 789903	2	+ . 4582 969125
3	+ . 2981 267117	3	+ . 1797 041902	3	− . 513 666132
4	− . 302291	4	− . 525 537119	4	− . 536 351675
5	− . 137 083309	5	− . 59 072312	5	+ . 51 392861
6	+ . 4 860041	6	+ . 18 633869	6	+ . 15 383196
7	+ . 2 908111	7	+ . 793379	7	− . 1 576396
8	− . 154931	8	− . 328454	8	− . 221467
9	− . 33092	9	− . 3373	9	+ . 24586
10	+ . 2111	10	+ . 3357	10	+ . 1848
11	+ . 235	11	− . 20	11	− . 229
12	− . 17	12	− . 22	12	− . 10
13	− . 1	13		13	+ . 1

n	$x = 16$	n	$x = 17$	n	$x = 18$
0	+2.072547 994853	0	+2.144590 530048	0	+2.207465 026150
1	+ . 73431 192124	1	+ . 68814 956009	1	+ . 56297 544762
2	+ . 530 186986	2	− . 4896 572023	2	− . 6785 900840
3	− . 1902 808806	3	− . 1416 889790	3	+ . 224 369981
4	− . 115 909367	4	+ . 324 725561	4	+ . 424 855909
5	+ . 101 637502	5	+ . 61 542290	5	− . 22 337471
6	+ . 488368	6	− . 12 403941	6	− . 13 273940
7	− . 2 322069	7	− . 1 103103	7	+ . 822659
8	+ . 42220	8	+ . 233416	8	+ . 210427
9	+ . 29361	9	+ . 10295	9	− . 14396
10	− . 900	10	− . 2578	10	− . 1988
11	− . 233	11	− . 53	11	+ . 147
12	+ . 9	12	+ . 19	12	+ . 12
13	+ . 1	13		13	− . 1

n	$x = 19$	n	$x = 20$	n	$x = 21$
0	+2.257591 301413	0	+2.300094 942939	0	+2.343461 124573
1	+ . 44914 240018	1	+ . 41648 766783	1	+ . 45877 187095
2	− . 3963 570294	2	+ . 629 608935	2	+ . 2981 978230
3	+ . 1474 096088	3	+ . 1343 038103	3	+ . 130 972807
4	+ . 155 405174	4	− . 206 301828	4	− . 346 118230
5	− . 75 159567	5	− . 58 055757	5	+ . 5 191833
6	− . 3 172325	6	+ . 8 122150	6	+ . 11 160477
7	+ . 1 805891	7	+ . 1 175494	7	− . 342124
8	+ . 20453	8	− . 161068	8	− . 187104
9	− . 24240	9	− . 13133	9	+ . 7359
10	+ . 126	10	+ . 1880	10	+ . 1890
11	+ . 206	11	+ . 91	11	− . 85
12	− . 3	12	− . 14	12	− . 13
13	− . 1	13		13	+ . 1

TABLE II:

$Ji_0(x)$ to 10 decimal places, for $x = 3(.1)10(1)22$, with even central differences up to $x = 10$.

TABLE III:

"Reduced" derivatives of $F(x)$ for $x = 10(1)21$ to 12 decimal places. Definition: Let $F^{(n)}(x)$ denote the n^{th} derivative of $F(x)$ with respect to x; then the n^{th} "reduced derivative" of $F(x)$ is defined as $A_n(x) = F^{(n)}(x)/n!$.

When x is small, $Ji_0(x)$ differences no better than $\log x$; for this reason, the function $F(x)$, which differences well in that region, is included in Table I. The logarithmic term can be obtained readily from available tables of natural logarithms.[4]

Table III may be used to obtain a value of $Ji_0(x)$ for x between 10 and 22. Thus let x be any point for which the reduced derivatives are given. Let $A_n(x)$ be denoted by A_n. Then by Taylor's expansion

$$F(x \pm t) = F(x) \pm tA_1 + t^2A_2 \pm t^3A_3 + \cdots$$

$$Ji_0(x \pm t) = F(x \pm t) - \log \tfrac{1}{2}(x \pm t)$$

Beyond $x = 22$, the asymptotic expansions yield very satisfactory results.

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[4] See for instance: Federal Works Agency "Table of Natural Logarithms," Vols. III and IV, distributed by the National Bureau of Standards, Washington, D. C.

AN ASYMPTOTIC EXPANSION OF $Ji_0(x) = \int_x^\infty \frac{J_0(t)}{t}\,dt$

By V. G. Smith[1]

Values of the above integral were required recently and the author computed a short table of the function. A more extensive table prepared under the auspices of the Mathematical Tables Project appears elsewhere in this issue.[2]

By analogy with the exponential, cosine and sine integrals it seems convenient to call this integral the "Bessel Integral of the First Kind and Order Zero." The symbol $Ji_0(x)$ is suggested by $Ei(x)$, $Ci(x)$ and $Si(x)$.

Using this notation and integrating by parts twice, we have

$$(1)\qquad \begin{aligned} Ji_0(x) &= \int_x^\infty \frac{tJ_0(t)\,dt}{t^2} = \int_x^\infty \frac{d[tJ_1(t)]}{t^2} \\ &= -\frac{J_1(x)}{x} - 2\int_x^\infty \frac{d[J_0(t)]}{t^2} \\ &= -\frac{J_1(x)}{x} + 2\frac{J_0(x)}{x^2} - 2^2\int_x^\infty \frac{J_0(t)}{t^3}\,dt. \end{aligned}$$

This process may be repeated more generally, giving

$$(2)\qquad \int_x^\infty \frac{J_0(t)}{t^n}\,dt = -\frac{J_1(x)}{x^n} + (n+1)\frac{J_0(x)}{x^{n+1}} - (n+1)^2\int_x^\infty \frac{J_0(t)}{t^{n+2}}\,dt.$$

By repeated application of this result,

$$(3)\qquad \begin{aligned} Ji_0(x) = {} & J_0(x)\left\{\frac{2}{x^2} - \frac{2^2 4}{x^4} + \frac{2^2 4^2 6}{x^6} - + \cdots + (-)^{n-1}\frac{2^2 4^2 \cdots (2n-2)^2(2n)}{x^{2n}}\right\} \\ & - J_1(x)\left\{\frac{1}{x} - \frac{2^2}{x^3} + \frac{2^2 4^2}{x^5} - + \cdots + (-)^{n-1}\frac{2^2 4^2 \cdots (2n-2)^2}{x^{2n-1}}\right\} \\ & + R. \end{aligned}$$

The remainder R is,

$$(4)\qquad R = (-)^n 2^2 4^2 \cdots (2n)^2 \int_x^\infty \frac{J_0(t)}{t^{2n+1}}\,dt$$

If M is the greatest value of $|\,J_0(t)\,|$ for $t > x$, then

$$(5)\qquad |R| < \frac{2^2 4^2 \cdots (2n-2)^2(2n)}{x^{2n}} \cdot M$$

It will be noted that the coefficient of M is the last term of the $J_0(x)$ series in equation (3).

[1] Associate Professor of Electrical Engineering, University of Toronto.

[2] Table of $Ji_0(x) = \int_x^\infty \frac{J_0(t)}{t}\,dt$ and Related Functions, Arnold N. Lowan, G. Blanch and M. Abramowitz.

This series will give $Ji_0(x)$ for values of $x > 10$ with an error less than four units in the fifth decimal place. For values of $x > 20$ the error is less than one unit in the ninth decimal place. By including only one-half of the smallest terms the error may be reduced. By applying the methods of J. R. Airey[3] it is probable that considerably greater accuracy could be obtained.

The usefulness of this series depends upon the availability of tables of $J_0(x)$ and $J_1(x)$. The British Association for the Advancement of Science Tables No. VI give values up to $x = 25$ and by auxiliary functions up to $x = 100$. The asymptotic series given in the accompanying paper[2] by Lowan, Blanch and Abramowitz requires only circular function tables but the series given here seems neater and easier to use when Bessel function tables are available.

[3] "The Converging Factor" in Asymptotic Series and the Calculation of Bessel, Laguerre and Other Functions. John R. Airey, Phil. Mag., Ser. 7, Vol. XXIV, Oct. 1937.

A METHOD FOR FINDING ROOTS OF ALGEBRAIC EQUATIONS

By Shih-ngb Lin

Principle

Let the given algebraic equation of mth degree

$$x^m + A_{m-1}x^{m-1} + \cdots + A_1x + A_0 = 0 \qquad \text{(I)}$$

be separated into two polynomials, one of the nth degree and the other $(m-n)$th degree as follows:

$$(x^n + b_{n-1}x^{n-1} \cdots + b_1x + b_0)(x^{m-n} + a_{m-n-1}x^{m-n-1} + \cdots + a_1x + a_0) + (r_{n-1}x^{n-1} + \cdots + r_1x + r_0) = 0 \qquad \text{(II)}$$

the last polynomial being the remainder terms.

The coefficients a and r for given divisor coefficients b can be found by performing synthetic divisions:

$$\begin{array}{l|l} -b_{n-1} - b_{n-2} \cdots b_1 - b_0 & 1 + A_{m-1} + A_{m-2} \cdots + A_{1+n} + A_n + A_{n-1} + \cdots \\ & \qquad + A_1 + A_0 \\ & \qquad \cdots \qquad\qquad -a_1b_0 - a_0b_0 \\ & \qquad -b_{n-2} \qquad\qquad - a_0b_1 \\ & \quad -b_{n-1} - (b_{n-1})(a_{m-n-1}) \\ \hline & 1 + a_{m-n-1} + a_{m-n-2} + \cdots + a_1 + a_0 + r_{n-1} + \cdots + r_1 + r_0 \end{array}$$

Thus,

$$a_{m-n-1} = A_{m-1} - b_{n-1}, \qquad (m-n-1)$$

$$a_{m-n-2} = A_{m-2} - b_{n-2} - (b_{n-1})(a_{m-n-1}), \qquad (m-n-2)$$

$$\cdots \text{etc.}$$

$$r_{n-1} = A_{n-1} - a_{n-1}b_0 - a_{n-2}b_1 - \cdots - a_1b_{n-2} - a_0b_{n-1} \cdots (n-1)$$

$$\cdots$$

$$r_1 = A_1 - a_1b_0 - a_0b_1 \cdots \qquad (1)$$

$$r_0 = A_0 - a_0b_0 \cdots \qquad (0)$$

In attempting to reduce the magnitudes of the coefficients r, a new set of divisor coefficients b' are so chosen that expressions:

$$[r_{n-1}] = A_{n-1} - a_{n-1}b'_0 - a_{n-2}b'_1 - \cdots - a_1b'_{n-2} - a_0b'_{n-1} = 0 \quad (n-1)'$$

$$[r_1] = A_1 - a_1b'_0 - a_0b'_1 = 0, \qquad (1)'$$

$$[r_0] = A_0 - a_0b'_0 = 0, \qquad (0)'$$

and b'_0 is then solved from equations $(0)'$, b'_1 from $(1)'$ etc.

Using the new set of divisor coefficients b' the process can be repeated, obtaining new coefficients a' and r'.

If the remainder coefficients r approach zero ultimately, the repetition of the process is said to be "convergent." The given equation is then exactly factored into polynomials of lower degree. If one of the polynomials is of quadratic degree, a pair of roots, real or complex, can be readily obtained.

Condition of Convergency and Choice of Initial Divisor

Write

$$r_0 = (0) - (0)' = a_0(b_0' - b_0) = a_0(\Delta b_0)$$

$$r_1 = (1) - (1)' = a_0(b_1' - b_1) + a_1(b_0' - b_0) = a_0(\Delta b_1) + a_1(\Delta b_0)$$

$$\cdots$$

$$r_{n-1} = (n-1) - (n-1)' = a_0(\Delta b_{n-1}) + a_1(\Delta b_{n-2}) + \cdots + a_{n-1}(\Delta b_0)$$

after one repetition of the process, we have

$$r_0' = a_0'(\Delta b_0)'$$

$$r_1' = a_0'(\Delta b_1)' + a_1(\Delta b_0)'$$

$$\cdots$$

$$r_{n-1}' = a_0'(\Delta b_{n-1})' + a_1'(\Delta b_{n-2})' + \cdots + a_{n-1}'(\Delta b_0)'$$

Since $\Delta b_0' = b_0'' - b_0' = (A_0)/a_0' - (A_0)/a_0$ by $(0)'$

$$= \frac{A_0}{a_0' a_0}(a_0 - a_0') = \left(\frac{-Aa_0}{a_0'}\right) b_0'$$

$$\Delta b_1' = b_1'' - b_1' = (A_1 - a_1' b_0'')/a_0' - (A_1 - a_1 b_0')/a_0 \quad \text{by} \quad (1)'$$

$$= A_1\left(\frac{1}{a_0'} - \frac{1}{a_0}\right) - \left[\left(\frac{a_1'}{a_0'}\right) b_0'' - \left(\frac{a_1}{a_0}\right) b_0'\right]$$

$$\doteq \left(\frac{A_1}{A_0}\right)\Delta b_0' - \left(\frac{a_1}{a_0}\right)\Delta b_0' \quad \text{if} \quad \left(\frac{a_1'}{a_1}\right)\left(\frac{a_0}{a_0'}\right) \doteq 1$$

$$\doteq \left(\frac{A_1}{A_0} - \frac{a_1}{a_0}\right)\Delta b_0'$$

Similarly, it can be shown that

$$\Delta b_2' \doteq \left[\frac{A_2}{A_0} - \frac{a_2}{a_0} - \frac{a_1}{a_0}\left(\frac{A_1}{A_0} - \frac{a_1}{a_0}\right)\right]\Delta b_0'$$

if $$\left(\frac{a_2'}{a_2}\right)\left(\frac{a_0}{a_0'}\right) \doteq 1 \quad \text{and} \quad \Delta b_{n-1}' \doteq B\Delta b_0'$$

(where B is a constant depending on coefficients A and a).

if $$\left(\frac{a_{n-1}'}{a_{n-1}}\right)\left(\frac{a_0}{a_0'}\right) \doteq 1$$

Therefore, if $\Delta b_0'$ approach zero ultimately, remainder coefficients r will vanish correspondingly provided coefficients a remain finite.

Judging from the expression for $\Delta b_0'$, it is seen that smallness of the ratio (b_0'/a_0') is favorable for small increment of b_0'. It is thus suggested that $b_0 = b_1 = b_2 = \cdots = b_{n-1} = 0$ be taken as coefficients for the initial divisor, obtaining

$$a_{m-n-1} = A_{n-1} \cdots , a_1 = A_{m-n-1}, \qquad a_0 = A_{m-n}$$

and

$$r_{n-1} = A_{m-n+1} \cdots , r_1 = A_1 , \qquad r_0 = A_0$$

for the first division, and also

$$\Delta b_0 = b_0' - b_0 = b_0'$$

so that we have

$$\Delta b_0' = \left(\frac{-\Delta a_0}{a_0'}\right) b_0' = \left(\frac{-\Delta a_0}{a_0'}\right) \Delta b_0$$

or

$$\left|\frac{\Delta b_0'}{\Delta b_0}\right| = \left|\frac{-\Delta a_0}{a_0'}\right|$$

Thus, it is seen that if $|-\Delta a_0/a_0'|$ for the first repetition of the process is much smaller than unity, $\Delta b'$ may approach zero ultimately.

Matrix for Numerical Computation of Coefficients

Though coefficients a and r for given coefficients A and b can be computed numerically through the usual process of synthetic division, much time and space can be saved by tabulation of coefficients obtained by expressions derived from simple rule of matrix, particularly where repetitions of the process are required.

Rule of Matrix. For the given equation (I) and any given set of divisor coefficients $b_{n-1} \cdots b_1 , b_0$; form the following matrix and denote columns by $\{1\}$, $\{2\} \cdots \{m\}$ and rows by $\{A\}$, $\{b\}$ and $\{a\}$:

Column / Row	1	2	3	...	n	$m-n-1$	$m-n$	...	...	$m-1$	m_1
A	A_{m-1}	A_{m-2}	A_{m-3}	...	...	A_{1-n}	A_n	...	...	A_1	A_0
b	$-b_{n-1}$	$-b_{n-2}$	$-b_{n-3}$	...	$-b_0$	0	0	0	0	0	0
a	a_{m-n-1}	a_{m-n-2}	a_{m-n-3}	...	...	a_1	a_0	0	0	0	0

The rule for expressing a and r can be written as:

$$a_{m-n-1} = A_{m-1} - b_{n-1} = \{1A\} + \{1b\}$$

$$a_{m-n-2} = A_{m-2} - b_{n-2} - b_{n-1}a_{m-n-1} = \{2A\} + \{2b\} + \{1b\} \times \{1a\}$$

$$a_{m-n-3} = A_{m-3} - b_{n-3} - b_{n-1}a_{m-n-2} - b_{n-2}a_{m-n-1} =$$

$$\{3A\} + \{3b\} + \{1b\} \times \{2a\} + \{2b\} \times \{1a\}$$

or in words: a in nth column = sum of A and $-b$ in nth column + cross product between row $\{b\}$ and row $\{a\}$ taking number of columns such that sum of the number of columns making product is equal to n.

An example may illustrate the rule:

Given equation:

$$x^8 + A_7x^7 + A_6x^6 + A_5x^5 + A_4x^4 + A_3x^3 + A_2x^2 + A_1x + A_0 = 0$$

Separate the given equation into two quartic factors:

$$(x^4 + b_3x^3 + b_2x^2 + b_1x + b_0)(x^4 + a_3x^3 + a_2x^2 + a_1x + a_0) + (r_3x^3 + r_2x^2 + r_1x + r_0) = 0$$

For any arbitrarily assigned divisor coefficients b_3, b_2, b_1 and b_0; $a_3 \cdots a_0$ and $r_3 \cdots r_0$ are found from the matrix:

A_7	A_6	A_5	A_4	A_3	A_2	A_1	A_0
$-b_3$	$-b_2$	$-b_1$	$-b_0$	0	0	0	0
a_3	a_2	a_1	a_0	0	0	0	0

Thus

$$a_3 = A_7 - b_3$$

$$a_2 = A_6 - b_2 - b_3a_3 \quad \text{see matrix } \{a\} \rightarrow$$

A_7	A_6
$-b_3$	$-b_2$
$\uparrow$	
$-a_3$	

$$a_1 = A_5 - b_1 - a_3b_2 - a_2b_3 \quad \text{see matrix } \{b\} \rightarrow$$

A_7	A_6	A_5
$-b_3$	$-b_2$	$-b_1$
$\nwarrow$	$\nearrow$	
a_3	a_2	

$$a_0 = A_4 - b_0 - a_3b_1 - a_1b_3 - a_2b_2 \quad \text{see matrix } \{c\} \rightarrow$$

A_7	A_6	A_5	A_4
$-b_3$	$-b_2$	$-b_1$	$-b_0$
$\nwarrow$	$\uparrow$	$\nearrow$	
a_3	a_2	a_1	

Similarly,

$$r_3 = A_3 - a_0b_3 - a_3b_0 - a_1b_2 - a_2b_1$$

$$r_2 = A_2 - a_0b_2 - a_2b_0 - a_1b_1$$

$$r_1 = A_1 - a_0b_1 - a_1b_0$$

$$r_0 = A_0 - a_0b_0$$

The divisor coefficients b' for repetition of the process can be derived from expressions for r by replacing b by b' and omitting the term involving a_0 which is taken as denominator. In the particular example we have:

$$b_0' = A_0/a_0$$
$$b_1' = (A_1 - a_1b_0')/a_0$$
$$b_2' = (A_2 - a_2b_0' - a_1b_1')/a_0$$
$$b_3' = (A_3 - a_3b_0' - a_1b_2' - a_2b_1')/a_0$$

Application of the Method

Example (I):
Take the quartic equation:

$$x^4 - 6x^3 + 47x^2 - 18x + 290 = 0$$

Separating the given equation into two quadratic factors, we have:

$$(x^2 + b_1x + b_0)(x^2 + a_1x + a_0) + (r_1x + r_0) = 0$$

Step (1): From the matrix

$$\begin{matrix} A_3 & A_2 & A_1 & A_0 \\ -b_1 & -b_0 & 0 & 0 \\ a_1 & a_0 & 0 & 0 \end{matrix}$$

obtain the following expressions:

$$a_1 = A_3 - b_1 \qquad a_0 = A_2 - b_0 - a_1b_1$$
$$r_1 = A_1 - a_1b_0 - a_0b_1 \qquad r_0 = A_0 - a_0b_0$$

and $$b_0' = (A_0/a_0) \qquad b_1' = (A_1 - a_1b_0')/a_0$$

Step (2): Numerical computation with

$$b_1 = b_0 = 0, \quad a_1 = A_3 = -6, \quad a_0 = A_2 = 47$$
$$r_1 = A_1 = -18, \quad \text{and} \quad r_0 = A_0 = 290.$$

Thus, for first repetition

$$b_0' = 290/47 = 6.17, \qquad b_1' = -(18 + 6 \times 6.17)/47 = 0.404$$

Repeating the division:

$$a_1' = -6 - 0.404 = -6.404$$
$$a_0' = 47 - 6.17 + (6.404)(0.404) = 43.42$$
$$r_1' = -18 + (6.17)(6.404) - (43.42)(0.404) = 3.97$$
$$r_0' = 290 - (43.42)(6.17) = 22.6$$

Note $-\Delta a_0/a_0' = (47 - 43.42)/43.42 = 0.0825$ for the first repetition.

The process can be further repeated by forming the following tabulation:

Repititions	b_0 A_0/a_0	b_1 $(A_1 - a_1b_0')/a_0$	a_1 $A_2 - b_1$	a_0 $A_3 - b_0 - a_1b_1$	r_1 $a_0(\Delta b_0)$	r_0 $a_0(\Delta b_1) + a_1(\Delta b_0)$	$\frac{-\Delta a}{a_0'}$
0	0	0	$-b$	47	−18	290	
1	6.17	0.404	−6.404	43.42	3.97	22.6	0.0825
2	6.69	0.572	−6.572	44.07	0.722	−485	0.0125
3	6.58	0.572	−6.572	44.18	−0.3433	−0.662	0.0001
4	6.565	0.571	−6.571	44.185	0	0	0

It is seen that $-\Delta a/a_0'$ approaches zero rapidly, so that final solution within engineering accuracy (accuracy of the slide rule used) can be easily attained: namely

$$r_{1,2} = -\left(\frac{0.571}{2}\right) \pm \sqrt{\left(\frac{.571}{2}\right)^2 - 6.565} = -.2855 \pm 2.545i$$

$$r_{3,4} = -\left(\frac{6.571}{2}\right) \pm \sqrt{\left(\frac{6.571}{2}\right)^2 - 44.185} = 3.2855 \pm 5.78i$$

as two pairs of roots for the given equation.

Example (2):

Given equation:

$$-x^6 - 2.2x^5 + 105.2x^4 - 140.6x^3 + 392.1x^2 + 273.9x + 65.1 = 0$$

Step (1): Separate into:

$$(x^4 + b_3x^3 + b_2x^2 + b_1x + b_0)(x^2 + a_1x + a_0) + (r_3x^3 + r_2x^2 + r_1x + r_0) = 0$$

From the matrix:

A_5	A_4	A_3	A_2	A_1	A_0
$-b_3$	$-b_2$	$-b_1$	$-b_0$	0	0
a_1	a_0	0	0	0	0

we get:

$$a_1 = A_5 - b_3 \qquad a_0 = A_4 - b_2 - a_1b_3$$

$$r_3 = A_3 - b_1 - a_1b_2 - a_0b_3 \qquad r_2 = A_2 - b_0 - a_1b_1 - a_0b_2$$

$$r_1 = A_1 - a_1b_0 - a_0b_1 \qquad r_0 = A_0 - a_0b_0$$

$$b_0' = A_0/a_0 \qquad b_1' = (A_1 - a_1b_0')/a_0$$

$$b_2' = (A_2 - b_0' - a_1b_1')/a_0 \qquad b_3' = (A_3 - b_1' - a_1b_2')/a_0$$

Step (2): Numerical computation by following tabulation:

Repititions	$\frac{b_0}{A_0/a_0}$	$\frac{b_1}{(A_1 - a_1 b_0')/a_0}$	$\frac{b_2}{(A_2 - b_0' - a_1 b_1')/a_0}$	$\frac{b_3}{(A_3 - b_1' - a_1 b_2')/a_0}$	$\frac{A_1}{A_5 - b_3}$	$\frac{a_0}{A_4 - b_2 - a_1 b_3}$	$\frac{-\Delta a_0}{a_0'}$
0	0	0	0	0	−2.2	105.2	
1	0.619	2.62	3.78	−1.282	−0.918	100.24	.0495
2	0.648	2.74	3.93	−1.395	−0.805	100.15	.0009
3	0.649	2.74	3.93	−1.400	−0.800	100.15	0

Thus, the given equation is separated into:

$$(x^4 - 1.4x^3 + 3.93x^2 + 2.74x + 0.649)(x^2 - 0.8x + 100.15) = 0$$

The given equation can also be factored into:

$$(x^2 + b_1x + b_0)(x^4 + a_3x^3 + a_2x^2 + a_1x + a_0) + (r_1x + r_0) = 0$$

after six repetitions, it is exactly factored into (within engineering accuracy)

$$(x^2 + 0.6x + 0.13)(x^4 - 2.8x^3 + 106.75x^2 - 204.27x + 500.9) = 0$$

The quartic equation in either case can also be solved by process as illustrated in example (I).

Equations Having Double or Nearly Double Factors

For equations having roots either complex or real whose modulus are widely separated, the forgoing method offers a rapid and convenient solution, for the process is in general highly convergent.

For equation such as $x^4 + 2x^3 + 3x^2 + 2x + 1 = 0$ (formed by $(x^2 + x + 1)^2 = 0$), the process is so slowly convergent that after fifteen repetitions, we get,

$$(x^2 + 0.927x + 0.894)(x^2 + 1.073x + 1.112) + (0.012x + 0.07) = 0$$

Another example for the solution of

$$x^4 - 5x^3 + 12.5x^2 - 16.5x + 10.5 = 0$$

The process starts off as if convergent but becomes divergent after eighth repetition when b_0' becomes greater than a_0'. It is important in practical application to determine whether the above process for a given equation would involve divergence or not. It is evident from discussion of the condition of convergency for the above process, that quartic equations having double or nearly double quadratic factors would always be unsuitable for the above process.

Sextic or higher order equations having triple or multiple complex roots are rare and $b_0' = a_0'$ only when $a_0' = (b')^n$ or when $b_0' = 1$.

Consider the quartic equation:

$$x^4 + A_3x^3 + A_2x^2 + A_1x + A_0 = 0,$$

which is always separable into

$$(x^2 + b_1x + b_0)(x^2 + a_1x + a_0) = 0.$$

With the remainder terms equal to zero, we have the following relations:

$$A_1 = A_3 - b_1 \tag{1}$$

$$a_0 = A_2 - b_0 - a_1b_1 \tag{2}$$

$$a_0b_1 = A_1 - a_1b_0 \tag{3}$$

$$a_0b_0 = A_0 \tag{4}$$

Eliminating a_1 and a_0 by (1), (3) and (4):

$$b_0 = \frac{A_1}{2(A_3 - b_1)} \pm \sqrt{\left[\frac{A_1}{2(A_3 - b_1)}\right]^2 - \frac{A_0b_1}{(A_3 - b_1)}} \tag{A}$$

Similarly, eliminating a_1 and a_0 by (1), (2) and (4):

$$b_0 = \frac{A_2 - b_1(A_3 - b_1)}{2} \pm \sqrt{\left(\frac{A_2 - b_1(A_3 - b_1)}{2}\right)^2 - A_0} \tag{B}$$

If the given quartic equation has double quadratic factors, b_0 in (A) and (B) must be real and single valued. Or, the second term in (A) and (B) must be vanish, giving

$$b_1^2 - A_3b_1 + \frac{A_1^2}{4A_0} = 0 \tag{A}'$$

and

$$b_1^2 - A_3b_1 + (A_2 - 2\sqrt{A_0}) = 0 \tag{B}'$$

(Note that A_2 and A_0 must be positive for quartic equations having complex roots.)

Since b_1 must also be real and single valued, thus

$$\frac{A_1^2}{4A_0} = \left(\frac{A_3}{2}\right)^2 = A_2 - 2\sqrt{A_0}$$

is the condition for quartic equations having double quadratic factors.

Let

$$\alpha = \frac{A_1^2}{4A_0}, \qquad \beta = A_2 - 2\sqrt{A_0} \qquad \gamma = \left(\frac{A_3}{2}\right)^2$$

Case (1) (a). If $\alpha = \beta = \gamma$, we have as exact solution:

$$a_0 = b_0 = +\sqrt{A_0}, \qquad a_1 = b_1 = \frac{A_3}{2}.$$

Case (1) (b). If $\alpha = \gamma$, $\beta < \gamma$, (even when β = negative number) we have as exact solution:

from (B)′: $$(a_1, b_1) = \frac{-A_3}{2} \pm \sqrt{\left(\frac{A_3}{2}\right)^2 - \beta},$$

and $$a_0 = b_0 = +\sqrt{A_0}$$

This can be easily shown, since a_1, b_1 are roots of $(B)''$ or $b_1^2 - A_3 b_1 + \beta = 0$, Thus,

$$a_1 + b_1 = A_3 \quad \text{and} \quad a_1 b_1 = \beta = A_2 - 2\sqrt{A_0}$$

with $a_0 = b_0 = +\sqrt{A_0}$, we have $a_0 + b_0 = 2\sqrt{A_0}$

Thus (1) (2) and (4) are satisfied.

With $$\alpha = \gamma, \quad \text{or} \quad A_1^2/4A_0 = \left(\frac{A_3}{2}\right)^2 \quad \text{or}$$

$$A_1 = A_3\sqrt{A_0} = (a_1 + b_1)\sqrt{A_0} = a_1 b_0 + a_0 b_1$$

(3) is also satisfied.

Case (2) (a). If

α slightly smaller than γ

β slightly greater than γ

The given quartic equation has nearly double quadratic factors. The two real roots found from (A)′ and (A) can be used as first approximation for (b_1, a_1) and (b_0, a_0) respectively. Exact solution of the given equation can be obtained by process of successive approximation illustrated in following numerical example:

Case (2) (b). If

β slightly smaller than γ

α slightly larger than γ

the given equation also has nearly double quadratic factors, but the first approximation for (b_1, a_1) and (b_0, a_0) should be found from (B)′ and (B) respectively.

Case (2) (c). If β has negative value (or much smaller than γ) but α greater or smaller than γ, the given equation can also be exactly separated by the process of successive approximation with the first approximation for (b_1, a_1) and (b_0, a_0) found from (B)′ and (B) respectively.

Case (3). If α, β and γ bear values widely separated, the quartic equation will have quadratic factors widely separated with b_0 much smaller than a_0 so that process of successive separation is highly convergent.

Thus, α, β and γ for given quartic equations can be used as separation test for quadratic factors.

Numerical example (3):

(a). Given equation: $x^4 + 2x^3 + 3x^4 + 2x + 1 = 0$

$$\alpha = (2)^2/(4)(1) = 1, \qquad \beta = 3 - 2\sqrt{1} = 1, \qquad \gamma = (\tfrac{2}{2})^2 = 1.$$

thys, $\alpha = \beta = \gamma = 1$, we have double quadratic factors.

$$b_1 = a_1 = A_3/2 = 1,$$

$$b_0 = a_0 = \sqrt{1} = 1.$$

The equation can be factored into $(x^2 + x + 1)^2 = 0$.

(b). $x^4 - 5x^3 + 12.5x^2 - 16.5x + 10.5 = 0.$

$$\gamma = \left(\frac{5}{2}\right)^2 = 6.25, \qquad \alpha = \frac{(16.5)^2}{4(10.5)} = 6.48 \text{ slightly greater than } \gamma.$$

$$\beta = 12.5 - 2\sqrt{10.5} = 6.02 \text{ slightly smaller than } \gamma.$$

Thus, from (B)′: $b_1^2 + 5b_1 + 6.02 = 0$, obtain $(a_1, b_1) = -2.02, -2.98$. From (B), obtain $(a_0, b_0) = 3.24$. The approximate factor for the given equation can be written as

$$(x^2 - 2.98x + 3.24)(x^2 - 2.02x + 3.24) = 0.$$

Multiplying out the above factors, we have

$$x^4 - 5x^3 + 12.5x^2 - 16.2x + 10.5 = 0.$$

It is seen that only coefficient for x differs slightly from that for the given equation (had A_1 been -16.21, or $\alpha = \gamma$, we would have exact solution).

To obtain the exact solution for $A_1 = -16.5$, the following process of successive approximation can be followed:

From (1), (2), (3) and (4), it is seen that (a_1, b_1); and (a_0, b_0) found in the first approximation satisfy expressions (1), (2) and (4), but not (3).

If $a_1 = -2.98$, $b_1 = -2.02$ $(a_0 = b_0 = 3.24)$ from (3), new b_1' would be -2.11, giving new $a_1' = -2.89$.

With this new b_1' and a_1', expression (B) would give (a_0', b_0') complex instead of real number. However, if $a_1 = -2.02$, $b_1 = -2.98$ $(a_0 = b_0 = 3.24)$.

From (3), $b_1' = -3.07$, $a_1' = -1.93$, substituting a_1', b_1' into (B), we get

$$(a_0', b_0') = 3.785, 2.775.$$

With $a_0' = 3.785$, $b_0' = 2.775$, $a_1' = -1.93$.

From (3), we get $b'' = -2.95$ showing that the correct value for b_1 is $-3.07 < b_1 < -2.98$.

(Note that had a_0' been chosen as 2.775, and $b_0' = 3.785$, $(a_1' = -1.93$ as before), (3) would give $b_1'' = -3.31$, showing that further repetition of the process would be divergent.)

Now, let

$$b_1 = \frac{-3.07 - 2.98}{2} = -3.01, \qquad a_1 = -1.99.$$

Substituting into (B) we have:

$$a_0 = 3.538 \qquad b_0 = 2.972,$$

From (3), $b_1' = -2.99$, ($a_1' = -2.01$) showing that correct b_1 should be $-3.01 < b_1 < -2.98$. Further repetition of the above process gives the exact solution:

$$(x^2 - 3x + 3)(x^2 - 2x + 3.5) = 0.$$

(c). $x^4 - 6x^3 + 47x^2 - 18x + 290 = 0.$

$$\gamma = (\tfrac{6}{2})^2 = 9. \qquad \alpha = (18)^2/4(290) = 0.279$$

$$\beta = 47 - 2\sqrt{290} = 12.94$$

Thus, the quadratic factors are highly separated and can be rapidly factored as shown in example (1).

(d). $x^4 - 5x^3 + 2.8x^2 - 16.5x + 10.5 = 0.$

$$\gamma = \left(\frac{5}{2}\right)^2 = 6.25. \qquad \alpha = \frac{(16.5)^2}{4(10.5)} = 6.48 \text{ slightly greater than } \gamma.$$

$$\beta = 2.8 - 2\sqrt{10.5} = -3.68 = \text{negative number.}$$

From (B)′: $b_1^2 + 5b_1 - 3.68 = 0$, giving

$$(a_1, b_1) = -5.651, 0.651$$

$$a_0 = b_0 = \sqrt{10.5} = 3.24 \quad \text{or from (B).}$$

Thus, the approximate quadratic factor can be written as:

$$(x^2 + 0.651x + 3.24)(x^2 - 5.651x + 3.24) = 0.$$

Multiplying out the approximate factors, we get

$$x^4 - 5x^3 + 2.8x^2 - 16.22x + 10.5 = 0.$$

Using similar process of successive approximation: with $a_1 = .651$, $b_1 = -5.651$, from (3), find $b_1' = -5.74$, $a_1' = 0.74$; and from (B), find $a_0' = 4.92$, $b_0' = 2.13$. Substituting into (3), $b_1'' = 3.68$.

Thus, correct b_1 is $-5.74 < b_1 < -5.651$.

By interpolation, let

$$b_1 = -5.651 + \left(\frac{5.74 - 5.651}{5.74 - 3.68}\right)(5.651 - 5.74)$$

$$= -5.654.$$

Thus, $a_1 = .654$, from (B), $a_0 = 3.474$, $b_0 = 2.926$. From (3), $b_1' = -5.29$.

In the course of computation, it is noted that correct b_1 is very close to -5.651.

Thus, let $b_1 = -5.651$, $a_1 = .651$, $b_0 = 3.24$. From (3), calculate $a_0' = (A_1 - a_1 b_0)/b_1 = 3.29$.

$$b_0' = \frac{10.5}{3.29} = 3.19.$$

From (B), $\left(\frac{A_2 - a_1 b_1}{2}\right)^2 - A_0 = \left(\frac{3.29 - 3.19}{2}\right)^2 = 0.0025$

or $\left(\frac{2.8 - a_1 b_1}{2}\right)^2 = 10.5025$ or $a_1 b_1 = -3.68$

giving $b_1 = -5.651$, $a_1 = .651$ within engineering accuracy.

The correct solution is

$$(x^2 - 5.651x + 3.19)(x^2 + 0.651x + 3.29) = 0.$$

It is seen here that while one pair of the roots is complex, the other two roots are real.

(e). $x^4 - 2.075x^3 + 0.67x^2 - 0.095x + 2.287 = 0.$

$\gamma = \left(\frac{2.075}{2}\right)^2 = 1.075 \qquad \alpha = (0.095)^2/4(2.287) = 0.001$

$\beta = 0.67 - 2\sqrt{2.287} = -2.356 =$ negative number.

From (B)′: $b_1^2 + 2.075b_1 - 2.356 = 0$, giving

$$(b_1, a_1) = 0.816, -2.891$$

$$(b_0, a_0) = \sqrt{2.287} = 1.515.$$

Thus, for first approximation, the solution is

$$(x^2 + 0.816x + 1.515)(x^2 - 2.891x + 1.515) = 0.$$

Multiplying out the approximate factors, we get

$$x^4 - 2.075x^3 + 0.67x^2 - 3.145x + 2.287 = 0.$$

It is seen that only coefficient of x differs again from the true value.

Following similar process of successive approximation, but in a tabular form, we have:

No. of Repetition	b_1 check formula $b_1 = \frac{A_1 - a_1 b_0}{a_0}$	a_1 $a_1 = A_2 - b_1$	b_0 $b_0, a_0 = \frac{A_2 - a_1 b_1}{2} \pm \sqrt{\left(\frac{A_2 - a_1 b_1}{2}\right)^2 - A_0}$	a_0
0	.816	−2.891	1.515	1.515
1	.878	−2.953	1.018	2.242
2	1.300	−3.375	0.484	4.725
check $b_1'' = 0.325$				
3 Interpolation, $b_1 = .878 + \frac{(1.3 - .878)^2}{1.3 - .325}$				
	=1.054	−3.229	0.675	3.395
check $b_1'' = 0.613$				

4 Interpolation $b_1 = .878 + \dfrac{(1.054 - .878)^2}{1.054 - .613}$

$=0.948 \qquad -3.023 \qquad 0.853 \qquad 2.684$

check $b_1'' = 0.924$

5 Mean: $b_1 = \dfrac{.948 + .924}{2}$

$=0.936 \qquad -3.011 \qquad 0.878 \qquad 2.609$

check $b_1'' = 0.973$

6 Mean: $b_1 = \dfrac{.948 + .936}{2}$

$=0.942 \qquad -3.017 \qquad 0.866 \qquad 2.644$

check $b_1'' = 0.948$

Further repetition gives the solution as:

$$(x^2 + 0.9435x + 0.863)(x^2 - 3.0185x + 2.653) = 0$$

Similar expressions for testing equations of eighth degree can be derived as follows:

Consider the equation

$$x^8 + A_7x^7 + A_6x^6 + A_5x^5 + A_4x^4 + A_3x^3 + A_2x^2 + A_1x + A_0 = 0$$

which can be factored into:

$$(x^4 + b_3x^3 + b_2x^2 + b_1x + b_0)(x^4 + a_3x^3 + a_2x^2 + a_1x + a_0) = 0$$

Since the remainder terms are zero, we have

$a_3 = a_7 - b_3$ (1) $\qquad a_2 = A_6 - b_2 - a_3b_3$ (2)

$a_1 = A_5 - b_1 - a_3b_2 - a_2b_3$ (3) $\qquad a_0 = A_4 - b_0 - a_3b_1 - a_1b_3 - a_2b_2$ (4)

$a_0b_0 = A_0$ (5) $\qquad a_0b_1 = A_1 - a_1b_0$ (6)

$a_0b_2 = A_2 - a_2b_0 - a_1b_1$ (7) $\qquad a_0b_3 = A_3 - a_3b_0 - a_1b_2 - a_2b_1$ (8)

From (7) and (5), eliminating a_0, we get

$$b_0^2 - \frac{(A_2 - a_1b_1)b_0}{a_2} + \frac{b_2}{a_2}A_0 = 0 \qquad (A_0)$$

For b_0 to be single valued, we get

$$\left(\frac{A_2 - a_1b_1}{2a_2}\right)^2 = \frac{A_0b_2}{a_2} \quad \text{or} \quad (A_0)'$$

$$A_2 - a_1b_1 = 2\sqrt{A_0a_2b_2}$$

Eliminating a_1 by (3), we get

$$b_1^2 - (A_5 - a_3b_2 - a_2b_3)b_1 + (A_3 - 2\sqrt{A_0a_2b_2}) = 0 \qquad (A_1)$$

Thus, for both b_0 and b_1 to be single valued, we have

$$\left(\frac{A_5 - a_3b_2 - a_2b_3}{2}\right)^2 = A_3 - 2\sqrt{A_0a_2b_2} \qquad (A_1)'$$

From (6) and (5), eliminating a_0, we get

$$b_0^2 - \frac{A_1}{a_1}b_0 + \frac{b_1}{a_1}A_0 = 0 \qquad (B_0)$$

For b_0 to be single valued, we get

$$\left(\frac{A_1}{2a_1}\right)^2 = \frac{b_1}{a_1}A_0 \qquad \text{or} \qquad (B_0)'$$

$$a_1b_1 = \frac{A_1^2}{4A_0}$$

Substituting a_1b_1 from $(B_0)'$ into $(A_0)'$, we get

$$a_2b_2 = \frac{(A_3 - A_1^2/4A_0)^2}{4A_0} = k_1 \qquad (A_0B_0)'$$

Eliminating a_2 by (2), we have

$$b_2^2 - (A_6 - a_3b_3)b_2 + k_1 = 0 \qquad (A_2)$$

For b_2 to be single valued,

$$\left(\frac{A_6 - a_3b_3}{2}\right)^2 = k_1 \qquad \text{or}$$

$$A_6 - a_3b_3 = 2\sqrt{k_1}$$

Eliminating a_3 by (1), we get

$$b_3^2 - A_7b_3 + (A_6 - 2\sqrt{k_1}) = 0 \qquad (A_3)$$

Let

$$\left(\frac{A_7}{2}\right)^2 = \gamma, \qquad A_6 - 2\sqrt{k_1} = \beta_1$$

Thus, for double quartic factors, we have

$$\left(\frac{A_7}{2}\right)^2 = \gamma = \beta_1 = A_6 - 2\sqrt{k_1} = A_6 - \frac{\left(A_3 - \frac{A_1^2}{4A_0}\right)}{\sqrt{A_0}} \qquad (I)$$

Eliminating a_0 by (4) and (5):

$$b_0^2 - (A_4 - a_3b_1 - a_1b_3 - a_2b_2)b_0 + A_0 = 0 \qquad (C_0)$$

For b_0 to be single valued, we have

$$A_4 - a_2b_1 - a_1b_3 - a_2b_2 = 2\sqrt{A_0} \qquad (C)'$$

For b_1 to be single valued, we have

$$a_1 = b_1 = \frac{A_1}{2\sqrt{A_0}} \quad \text{by} \quad (B_0)'$$

Substituting into $(C_0)'$

$$A_4 - (a_3 + b_3)\frac{A_1}{2\sqrt{A_0}} - a_2b_2 = 2\sqrt{A_0}$$

or

$$a_2b_2 = A_4 - 2\sqrt{A_0} - \frac{A_1A_7}{2\sqrt{A_0}} = k_2 \qquad (B_0C_0)'$$

Thus,

$$b_2^2 - (A_6 - a_3b_2)b_2 + k_2 = 0 \qquad (C_2)$$

For b_2 single valued,

$$A_6 - a_3b_3 = 2\sqrt{k_2}$$

Eliminating a_3,

$$b_3^2 - A_7b_3 + (A_6 - 2\sqrt{k_2}) = 0 \qquad (C_3)$$

Let $$\beta_2 = A_6 - 2\sqrt{k_2},$$

for double quartic factors, we have also:

$$\left(\frac{A_7}{2}\right)^2 = \gamma = \beta_2 = A_6 - 2\sqrt{k_2} \qquad \text{(II)}$$

From (8) and (5), eliminating a_0, we have

$$b_0^2 - \left(\frac{A_3 - a_1b_2 - a_2b_1}{a_3}\right)b_0 + \frac{b_3}{a_3}A_0 = 0 \qquad (D_0)$$

For b_0 single valued, we get

$$A_3 - a_1b_2 - a_2b_1 = 2\sqrt{A_0a_3b_3} \qquad (D_0)'$$

For $$a_1 = b_1 = \frac{A_1}{2\sqrt{A_0}} \text{ by } (B_0)'$$

$$a_2 = b_2 = \sqrt{k} \text{ if } k = k_1 = k_2 \text{ by } (A_0B_0)' \text{ and } (B_0C_0)'$$

$$a_3 = b_3 = \left(\frac{A_7}{2}\right) \text{ by (1).}$$

Thus, for case of double quartic factors, we have, in addition to (I) and (II)

$$k = k_1 = k_2 \quad \text{and}$$

$$A_3 - \frac{A_1}{\sqrt{A_0}}\sqrt{k} = A_7\sqrt{A_0} \tag{D_1}$$

Substituting $a_1 = b_1 = A_1/2\sqrt{A_0}$, $a_2 = b_2 = \sqrt{k}$, $a_3 = b_3 = A_7/2$ into (3), we get

$$A_5 - \frac{A_1}{\sqrt{A_0}} = A_7\sqrt{k} \tag{D_1)'}$$

Eliminating $\sqrt{k}$ by (D_1) and $(D_1)'$, we get

$$\alpha_1 = \frac{A_5}{A_7} - \frac{A_1}{A_7\sqrt{A_0}} = \alpha_2 = \frac{A_3}{A_1}\sqrt{A_0} - \frac{A_7}{A_1}A_0 = \sqrt{k} \tag{III}$$

Thus, for a given equation of eighth degree, we have:

Case (1). If equations (I), (II) and (III) are all satisfied, we have double quartic factors

$$a_0 = b_0 = \sqrt{A_0}, \qquad a_1 = b_1 = \frac{A_1}{2\sqrt{A_0}}$$

$$a_2 = b_2 = \sqrt{k}, \qquad a_3 = b_3 = \frac{A_7}{2}$$

Case (2). If β_1 approximately equals to β_2 and α_1 approximately equals to α_2 and $\sqrt{k}$ we have approximately double quartic factors.

Case (3). If values of β_1 and β_2, α_1 and α_2 are widely separated, we have roots of the given equation also widely separated so that process of separating the given equation similar to that illustrated in examples (1) and (2) will in general be highly convergent.

Example (4)

(a). $x^8 + 8x^7 + 32x^6 + 80x^5 + 136x^4 + 160x^3 + 128x^2 + 64x + 16 = 0$

$$\gamma = \left(\frac{8}{2}\right)^2 = 16. \qquad k_1 = \left[128 - \frac{64^2}{4(16)}\right]^2 \Big/ 4(16) = 64$$

$$\beta_1 = 32 - 2\sqrt{64} = 16. \qquad k_2 = 136 - 8 - 64 = 64$$

$$\beta_2 \pm 16 = \beta_1 = \gamma. \qquad \alpha_1 = \frac{80}{8} - \frac{64}{8\sqrt{16}} = 8 = \sqrt{k}$$

$$\alpha_2 \pm \frac{160}{64}\sqrt{16} - \frac{8}{64}(16) = 8 = \sqrt{k} = \alpha.$$

Thus, we have double quartic factors

$$a_0 = b_0 = \sqrt{16} = 4, \qquad a_1 = b_1 = \frac{64}{2\sqrt{16}} = 8$$

$$a_2 = b_2 = \sqrt{k} = 8, \qquad a_3 = b_3 = 8/2 = 4$$

or the given equation can be factored into

$$(x^4 + 4x^3 + 8x^2 + 8x + 4)^2 = 0$$

(b). $x^8 + 2.61x^7 + 30.52x^6 + 71.62x^5 + 229.2x^4 + 224.8x^3 + 668.2x^2 - 352x + 573 = 0$

$$\gamma = \left(\frac{2.61}{2}\right)^2 = 1.70 k_1 = \left[668.2 - \frac{352^2}{4(573)}\right]^2 \Big/ (4)(573) = 164.4$$

$$\beta_1 = 30.52 - 2\sqrt{164.4} = 4.92 > \gamma$$

$$k_2 = 229.2 - 2\sqrt{573} + \frac{(2.61)(352)}{2\sqrt{573}} = 201.3$$

$$\beta_2 = 30.52 - 2\sqrt{201.3} = 2.16 > \gamma;$$

$$\alpha_1 = \frac{71.62}{2.61} + \frac{352}{2.61\sqrt{573}} = 33.0$$

$$\alpha_2 = \frac{224.8}{-352}\sqrt{573} + \frac{2.61}{352}(573) = -11.2$$

Thus we have roots of the equation widely separated. The equation can be factored into $(x^2 - 0.821x + 0.739)(x^6 + 3.431x^5 + 32.6x^4 + 95.84x^3 + 283.8x^2 + 586.7x + 776) = 0$.

By five repetitions, having $\left|\frac{\Delta a_0}{a_0'}\right| = 0.065$ for the first repetition of the process.

(c). $x^8 - 3.012x^7 + 3.225x^6 + 1.021x^5 + 5.986x^4 - 21.887x^3 + 8.11x^2 + 5.901x + 23.889 = 0$

We have $\gamma = 2.27$, $k_1 = 0.637$, $\beta_1 = 1.63$, $k_2 = .226$

$\beta_2 = 2.275$, $\alpha_1 = 0.065$, $\alpha_2 = -5.9$

Thus, the roots are also separated. It can be convergently factored into

$(x^4 - 2.075x^3 + 0.67x^2 - .095x + 2.287)(x^4 - .937x^3 + .61x^2 + 3.012x + 10.44) = 0$

By nine repetitions with $\left|\frac{-\Delta a_0}{a_0'}\right| = 0.227$ for the first repetition of the process.

It is believed that exact solution for equations having approximately equal factors can be solved in a similar way as illustrated in example (3), though the process may be too complicated for practical purpose.

Conclusion. The present method offers a convenient solution for all types of quartic equations having complex roots. Equations higher than sixth degree can also be rapidly solved if the roots are widely separated. It would be inconvenient to use the present method only when the roots are such that all quadratic factors of the given equation bear constant term b_0 very close to unity.

For such particular case, transformation of the given equation by simple root diminishing process before direct application of the method may be necessary.

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THE QUINTIC "HYPERNOM" FOR THE EQUATION

$$x^5 + Ax^3 + Bx^2 + Cx + D = 0$$

A Graphical Method for Finding the Roots of Polynomial Equations through the Fifth Degree

By Douglas P. Adams

The Aspect of the Chart

Figure (1)[1]. The chart is designed to give directly the real roots of the equation

(1) $$x^5 + Ax^3 + Bx^2 + Cx + D = 0 \qquad A, B, C, D \text{ not all zero.}$$

where the coefficients vary from -10 to $+10$ as follows:

A varies along the outer graduations of the left hand scale
B " " " inner " " " " " "
C " " " outer " " " " right " "
D " over " D-family of curves.

Root values of x from 0 to 10 appear in the upper and lower margins.

Near the top, bottom and center of the chart, each of the thirty-two vertical dashed lines (L-lines) carries its italic serial number subscribed with a semicircular marker. Near the top, bottom and center of the chart, each of the thirty-two vertical solid lines (M-lines) carries its unitalicized serial number superscribed with a triangular marker.

Figure 2 shows in simplified form how these two vertical grids would be used to create a curve S in the case of the quintic with coefficients $A = -7$, $B = 1$, $C = 2$.

Draw the straight lines CA and CB across the chart.

Let P_j be the point where the vertical dashed line L_j cuts CA.

Let N_j be the straight line through P_j parallel to CB.

Let Q_j be the point where N_j cuts the vertical solid line M_j.

The points Q_j determine the smooth, uninflected curve S. We shall see later that the intersections R_1, $R_2 \cdots$ of S with a specific D-curve yield the root values recorded directly above and below in the margin of the chart, but no D-curve has been drawn in Figure 2.

Figure 3 illustrates a simple method for carrying out this procedure. Secure an 8 x 10 chart on a small drawing board by a convenient weight and in such a position that BC lies in the prime direction. When the ruling edge passes through P_j on L_j, tick off Q_j on M_j. This determination of the S-curve by its points Q_j soon becomes accurate and rapid, reference to the identifying serial

[1] All of the charts appearing in this article have been reduced from a large master chart by photostatic methods. The distortions attendant in such reductions prohibit subsequent accurate use of the chart.

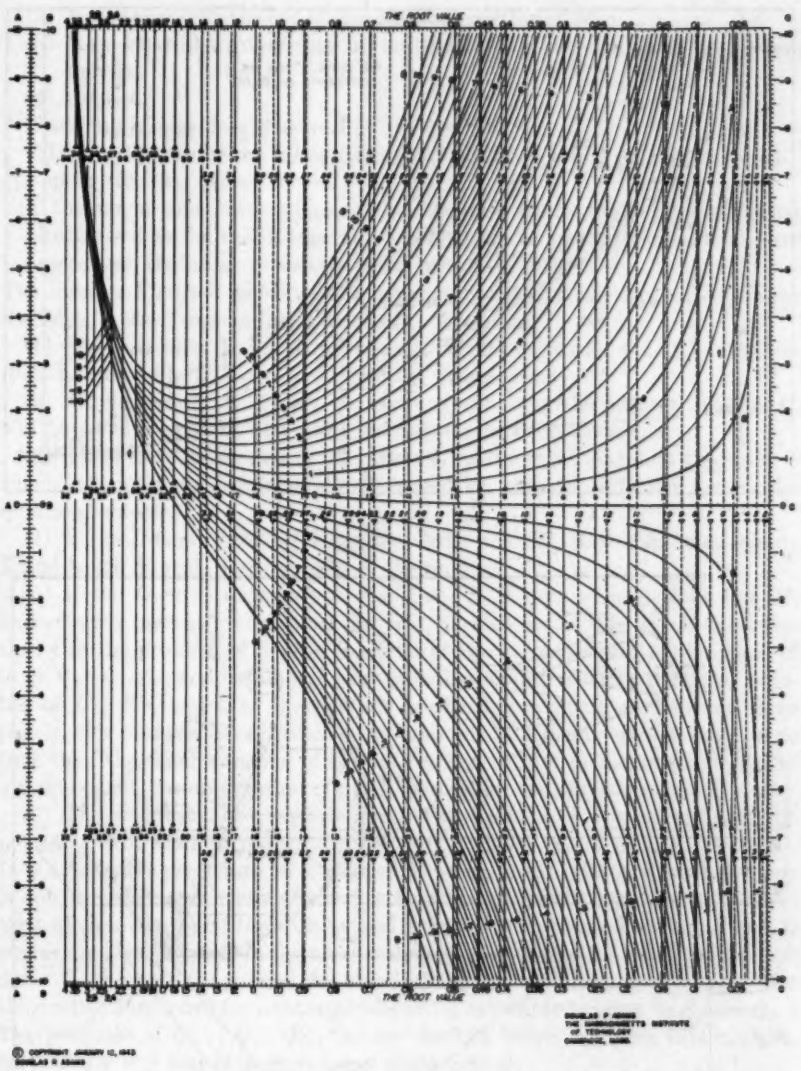

FIG. 1. The Quintic Hypernom for the equation

$$x^5 + A \cdot x^3 + B \cdot x^2 + C \cdot x + D = 0$$

numbers being relatively infrequent. Thus if Q_{17} has just been ticked off from P_{17}, the ruling edge either does or does not obscure P_{18}. If it does not

(1) Note P_{17} and fix a sharp pencil on P_{18}

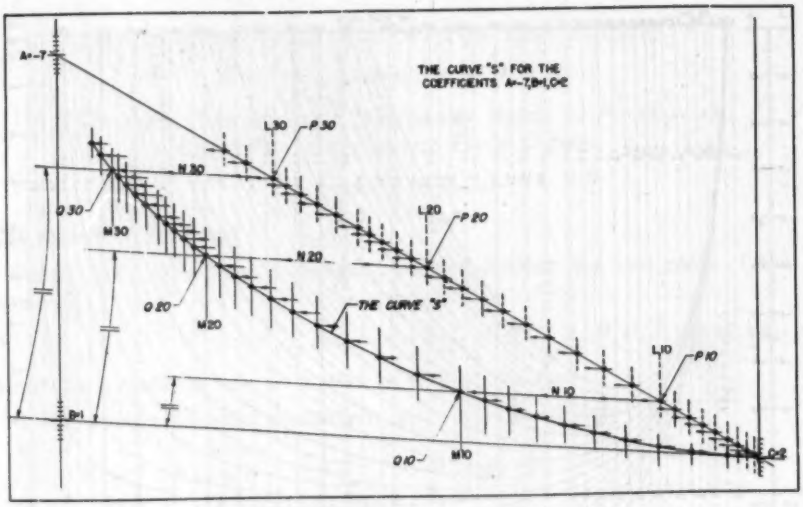
THE CURVE "S" FOR THE
COEFFICIENTS A=-7, B=1, C=2
A=-7
L30
P30
N30
D30
M30
L20
P20
N20
D20
M20
THE CURVE "S"
L10
P10
N10
D10
M10
B=1
C=2

FIG 2

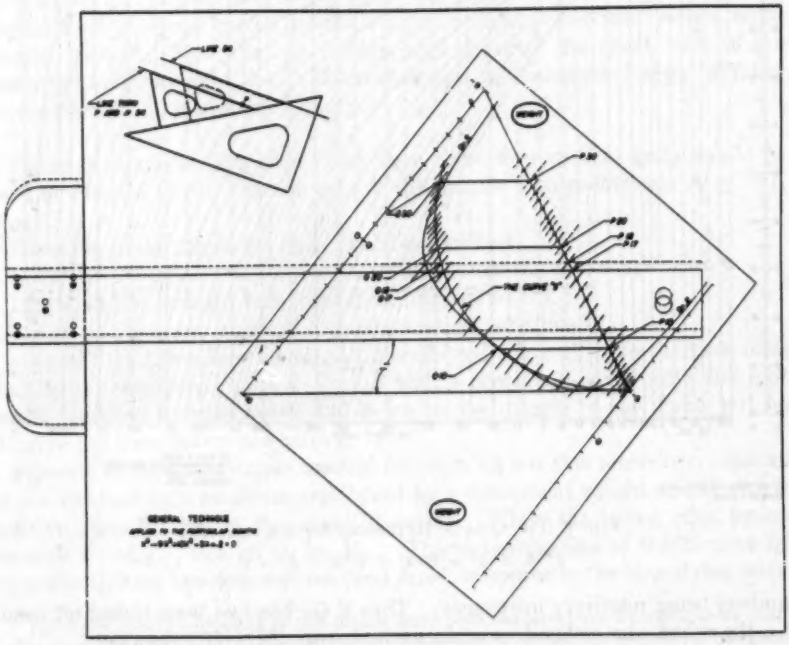

FIG. 3

(2) Note Q_{17} and fix the eye on the adjacent portion of M_{18}

(3) Now move the ruling edge by touch to lie against the pencil on P_{18} and tick Q_{18}

(4) Repeat.

If the ruling edge does obscure P_{18}

(1) Note P_{17}, move the ruling edge and fix a sharp pencil on P_{18}

Repeat (2), (3), (4) as above.

If the upper and lower edges of the T-square are mutually parallel, either can be used as the ruling edge and a choice can be made between the above procedures, the latter procedure being perhaps a little easier.

The inset of Figure 3 shows how a moving triangle can be used to determine the lines N_j if a T-square is not available. Remarks:

(1) Since the lines N_j are parallel to CB, within the wedge CA-CB, it is clear that the curve S will always lie within this wedge area.

(2) Moving from right to left, the curve S will always be tangent to CB at C, will bow out toward B, and will subsequently proceed toward A.

(3) These facts plus a plot of Q_{10}, Q_{20}, Q_{30} yield a rapid sketch of the curve S which will often be helpful and will indeed be entirely sufficient for certain quintic problems.

(4) If the D-curve does not enter the wedge CA-CB, there can be no points R and hence no real, positive roots of the quintic.

(5) If the D-curve enters the wedge CA-CB, it does so near some line M_e and emerges presently near some line M_k, thereby restricting the M-lines pertinent to the problem of finding R-points in this neighborhood to the panel of lines $M_e \cdots M_k$, and entailing a corresponding reduction in the pertinent number of L's, P's, and Q's. For quintic coefficients creating as many as three panels, this observation of panel boundaries would consume considerably more time than the rapid insertion of the entire S-curve, but in the frequent case of a single panel, the observation of panel boundaries usually saves time.

(6) The weighting of a smooth S-curve by means of a scroll through the points Q_j will tend to reduce the effect of random errors in their locations. It is a desirable procedure on a large chart where great accuracy is sought, but is not necessary on a chart of size 8 x 11, where it can be replaced by a chord method or a chord and tangent method. In the latter case, since the S and D curves are always gentle in the narrow strip $M_{j-1}M_{j+1}$, with a sharp hard pencil draw T_j through Q_j parallel to chord $Q_{j-1}Q_{j+1}$ and interpolate on D. (See Illustrative Problem 5 for treatment when D is ostensibly tangent to S near Q_j.) The positions of Q_{j-1}, Q_j, Q_{j+1} can be checked before applying this method, particularly if S and D meet at a low angle near Q_j.

(7) The life of a chart will be long if the constructions described above are carried out lightly with a sharp drawing pencil on a protective sheet of thin tracing paper.

The Use of the Chart to Find the Real Roots of a Quintic Equation

It is generally accepted that Horner's method or Newton's method for the accurate evaluation of real polynomial roots is extremely satisfactory once the

difficult problem of determining the existence and general magnitude of such roots has been solved. The chart quickly furnishes the first two figures of each real root and hence is a natural foundation for either of these methods.

To find all the real roots of the quintic

$$x^5 + Gx^4 + A'x^3 + B'x^2 + C'x + D' = 0 \tag{1}$$

I. Diminish the roots of (1)[2] by the quantity $-G/5$ to place the equation in the standard form

$$x^5 + Ax^3 + Bx^2 + Cx + D = 0 \tag{2}$$

II. Locate points R as described earlier. In either marginal scale, directly above or below R_ϱ, read the positive real root x_ϱ of (2) associated with R_ϱ. There cannot be more than four such roots. There may be none.

III. Reverse the signs of B and D and repeat. The real roots x'_ϱ so obtained are, with signs reversed, the negative real roots x_ϱ of equation (2). There cannot be more than four such roots. There may be none. If $D \neq 0$, there will always be at least one real root, either positive or negative. There may be five such real roots.

IV. Each real root x_ϱ of (2) is, when increased by $-G/5$, a real root of (1). Now employ Horner's method on (1) (without slide rule) if greater accuracy is desired than the chart has afforded.

Illustrative Problem 1

Figure 4. Find the general distribution and magnitude of the real roots of the quintic

$$f(x) = x^5 - 9x^3 + 10x^2 - 3x + .2 = 0 \tag{1}$$

The quintic is already in the fundamental form (2) and hence requires no diminution of the roots. In Figure 4, the complete curve S has been inserted and the locations of the four positive roots indicated on it. On reversing the signs of B and D the wedge CA-CB becomes very narrow. The curve $D = -0.2$ cuts it in the narrow panel M_{30}, M_{31}, M_{32}. The negative fifth root so obtained there is indicated. The chart values which were actually read off on inspection and the values given by Horner's method are compare below.

Chart: (0.095), (0.34), (0.75), (2.2) $\cdots$ (−3.5)
Horner's method: (0.0932), (0.3571), (0.7592), (2.2722), (−3.4820)

[2] (a) The operator who plans to use the chart repeatedly should be familiar with synthetic division and methods for diminishing polynomial roots. For a concise treatment of these subjects, Horner's method, and for further material helpful on the subject of polynomial roots see "College Algebra," Henry B. Fine, Ginn & Co., Boston, pp. 166–169, pp. 425–492. (b) The slide rule is efficient in this paper unless otherwise indicated. It is especially helpful with synthetic division, diminishing of polynomial roots, and Horner's method.

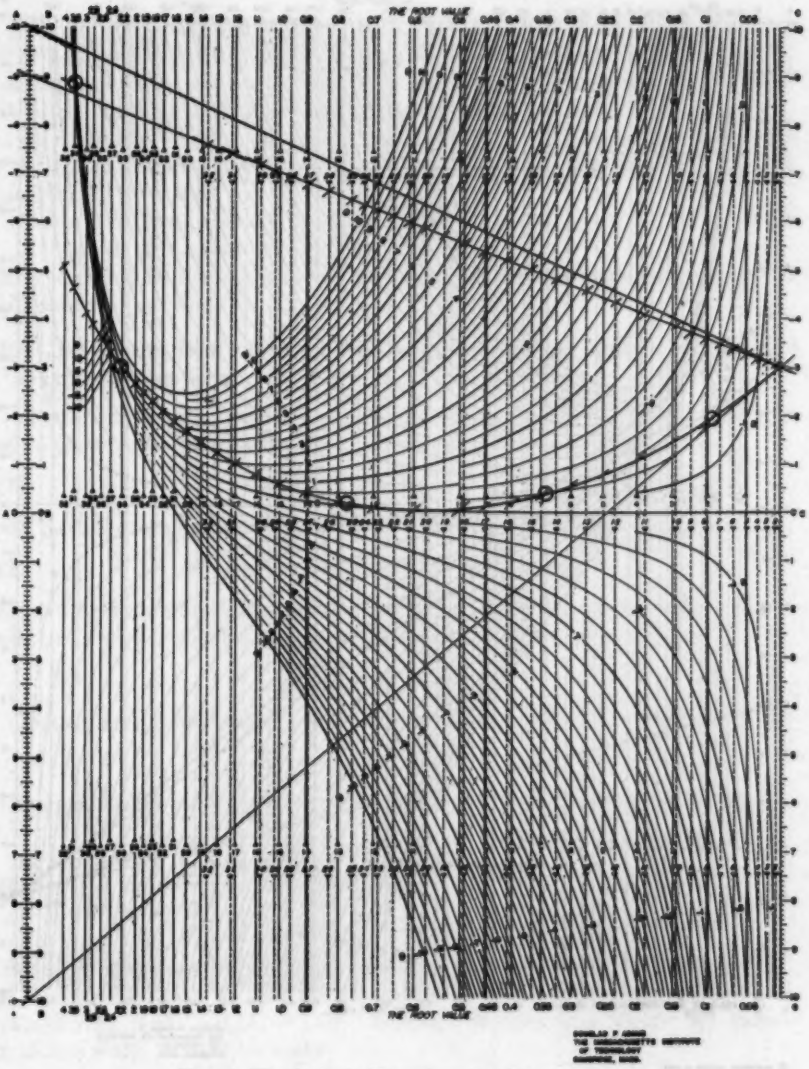

FIG. 4. The Quintic Hypernom for the equation

$$x^5 + A \cdot x^3 + B \cdot x^2 + C \cdot x + D = 0$$

Illustrative Problem 2

Figure 5. Find the value to five figures of the real roots of the quintic

(1) $$x^5 + 5x^4 + 7x^3 - 3x^2 - 4x + 3 = 0$$

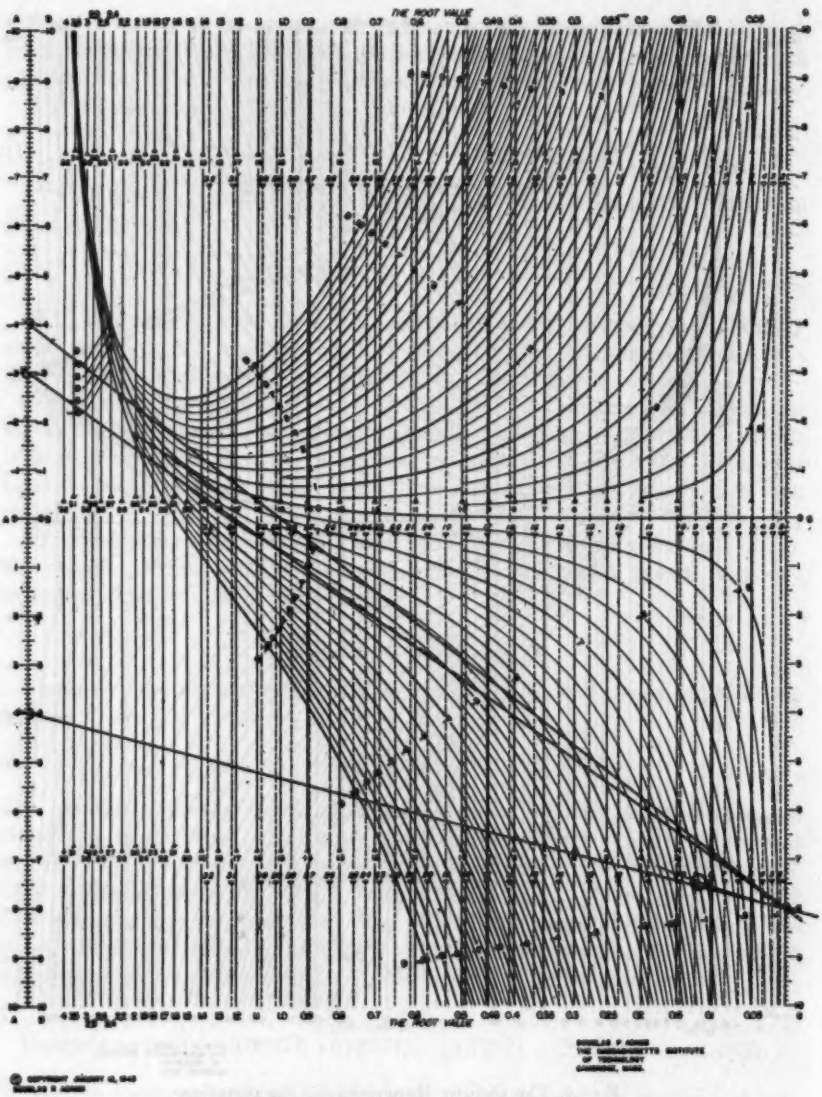

FIG. 5. The Quintic Hypernom for the equation

$$x^5 + A \cdot x^3 + B \cdot x^2 + C \cdot x + D = 0$$

Diminishing the roots of the cubic by $-5/5 = (-1)$, we obtain the quintic in the standard form

$$x^5 - 3x^3 - 4x^2 + 8x + 1 = 0 \tag{2}$$

The narrow wedge CA-CB, ($A = -3$, $B = -4$, $C = 8$) is cut by the curve $D = 1$ in the panel $M_{18} - M_{25}$. On locating the points $Q_{18} - Q_{25}$, it is apparent on inspection that an S-curve through them would never be cut by the curve $D = 1$. Thus the quintic has no real positive roots.

On reversing the signs of B and D, the curve $D = -1$ is seen to cut the wedge CA-CB in a single narrow panel between M_2 and M_3. The points Q_1, Q_2, Q_3 yield a root $x = 0.13$.

Hence -0.13 is a root of (2) and
-1.13 is a root of (1)

Now applying Horner's method, we quickly evaluate the single real root of this quintic as $x = -1.1294$.

The Use of the Chart to Find the Real Roots of the Quartic, Cubic, and Quadratic Equations

A quartic equation can be regarded as a quintic equation from which a root of value zero has been factored. The quartic

$$x^4 + Ax^2 + Bx + C = 0 \tag{1}$$

will have for its quintic of reference

$$x^5 + Ax^3 + Bx^2 + Cx + D = 0 \qquad D = 0 \tag{2}$$

Hence the procedure is the same as before, employing the curve $D = 0$. A similar process is applicable to the cubic and quadratic equations.

Illustrative Problem 3

Figure 6. Find the values to four places of the real roots of the quartic

$$x^4 - 5x^3 + 2x^2 + 9x + 1 = 0$$

Diminishing the roots by 1.25 we place this quartic in the standard form.

$$x^4 - 7.37x^2 - 1.61x + 8.06 = 0$$

Hence, for the purposes of the chart,

$$A = -7.37 \quad B = -1.61 \quad C = 8.06 \quad D = 0$$

The roots of the reduced quartic, by chart, are 1.01, 2.60, -1.36, -2.25
When increased by 1.25, they are the roots of the original equation and have the values 2.26, 3.85, -0.11, -1.00. Horner's method now yields the values: 2.2540, 3.8608, $-.1149$, -1.0000.

Illustrative Problem 4

Figure 7. To find the values of all of the real roots of the cubic

$$x^3 + 3x^2 - 4x + 4 = 0$$

We shall treat the cubic as we did the quartic above, namely as a quintic, with $C = 0$, $D = 0$

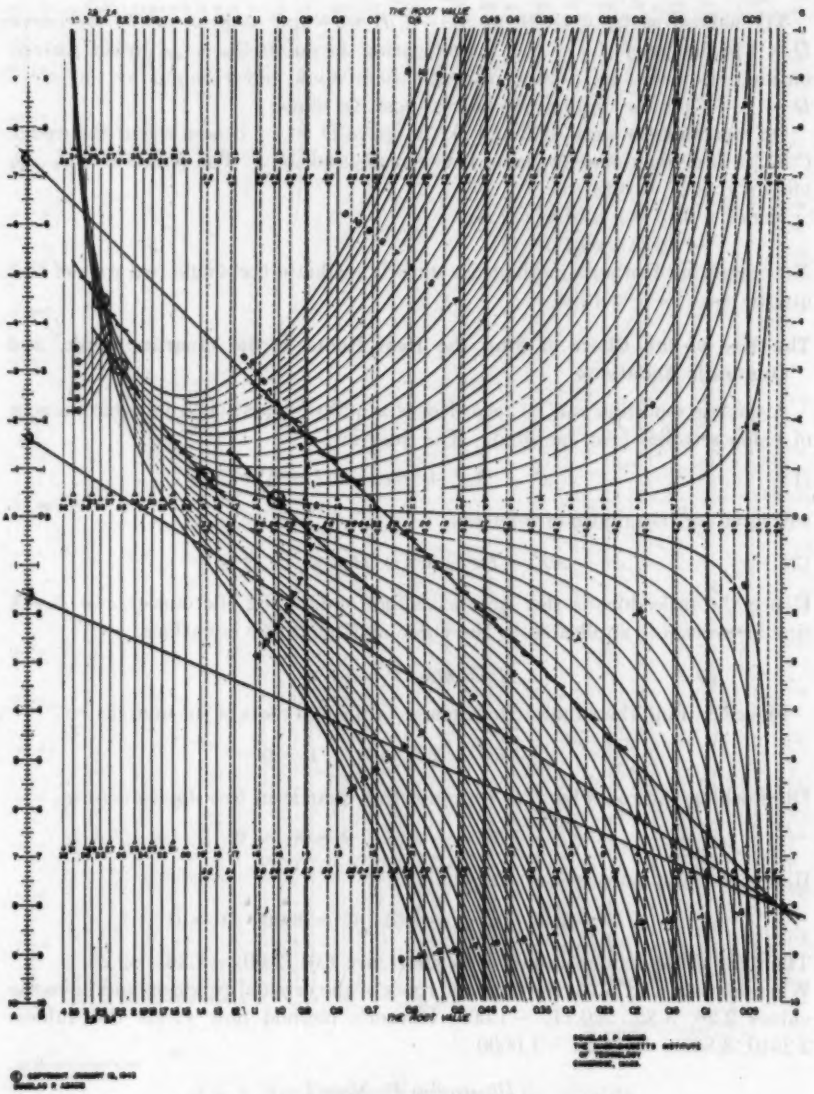

FIG. 6. The Quintic Hypernom for the equation
$x^5 + A \cdot x^3 + B \cdot x^2 + C \cdot x + D = 0$

Diminishing the roots of the cubic by -1, we place it in the standard form:

$$x^3 - 7x + 10 = 0$$

where we now have, for the purposes of the chart,

$$A = -7, \quad B = 10, \quad C = 0, \quad D = 0$$

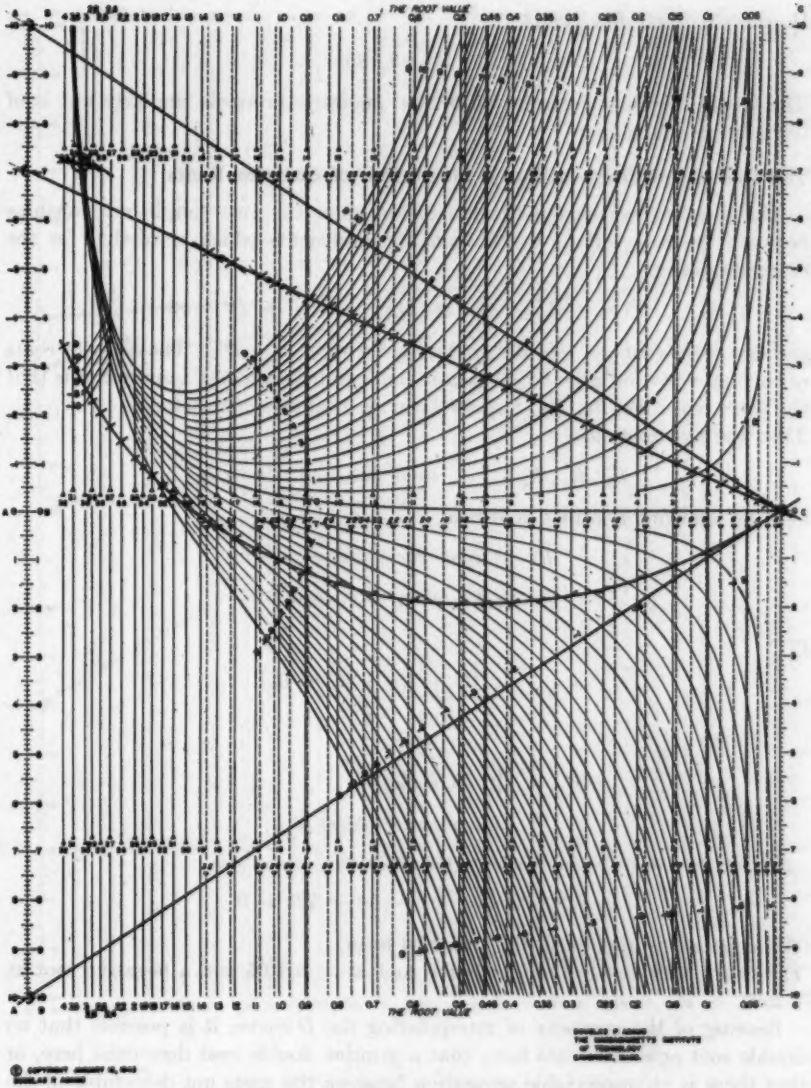

FIG. 7. The Quintic Hypernom for the equation

$$x^5 + A \cdot x^3 + B \cdot x^2 + C \cdot x + D = 0$$

The chart yields the single real root value for this equation

$$x = 3.20$$

The root value for the original equation will accordingly be

$$x = -4.20$$

Horner's method now yields

$$x = -4.185$$

The use of the chart for solution of the quadratic proceeds similarly but is of trivial practical value.

The Use of the Chart to Find a Single Pair of Complex Roots

The most expeditious method depends upon the two simple relationships between the roots of the quintic and its coefficients which state that for the original quintic

$$x^5 + Gx^4 + A'x^3 + B'x^2 + C'x + D' = 0$$

the sum of the roots $= -G$, the product of the roots $= -D'$. The complex roots of such an equation arise by conjugate pairs and the present assumption is that the three real roots R_1, R_2, R_3 have been ascertained accurately.
Then the five roots are

$$R_1, R_2, R_3, \qquad (a + bi), \qquad (a - bi)$$

and the following relations hold true

$$R_1 + R_2 + R_3 + 2a = -G$$
$$R_1 \cdot R_2 \cdot R_3 \cdot (a^2 + b^2) = -D'$$

or

$$a = -\left\{\frac{G + R_1 + R_2 + R_3}{2}\right\}$$

$$b = \sqrt{\frac{-D'}{R_1 R_2 R_3} - a^2}$$

Illustrative Problem 5

Figure 8. Find all the roots of the quintic

$$x^5 - 3x^3 + 4x^2 - 5x + 2.9 = 0$$

The equation is already in the standard form.
The chart indicates a double positive root at $x = 1.05$ and a negative root at -2.40.

Because of the necessity of interpolating the D-curve, it is possible that no double root actually exists here, that a genuine double root does exist here, or that there is an appreciable separation between the roots not detectable in the chart.

We observe by slide rule synthetic division that

$$f(.5) = +(1.06)$$
$$f(1) = -(0.1)$$
$$f(1.5) = +(0.0865)$$

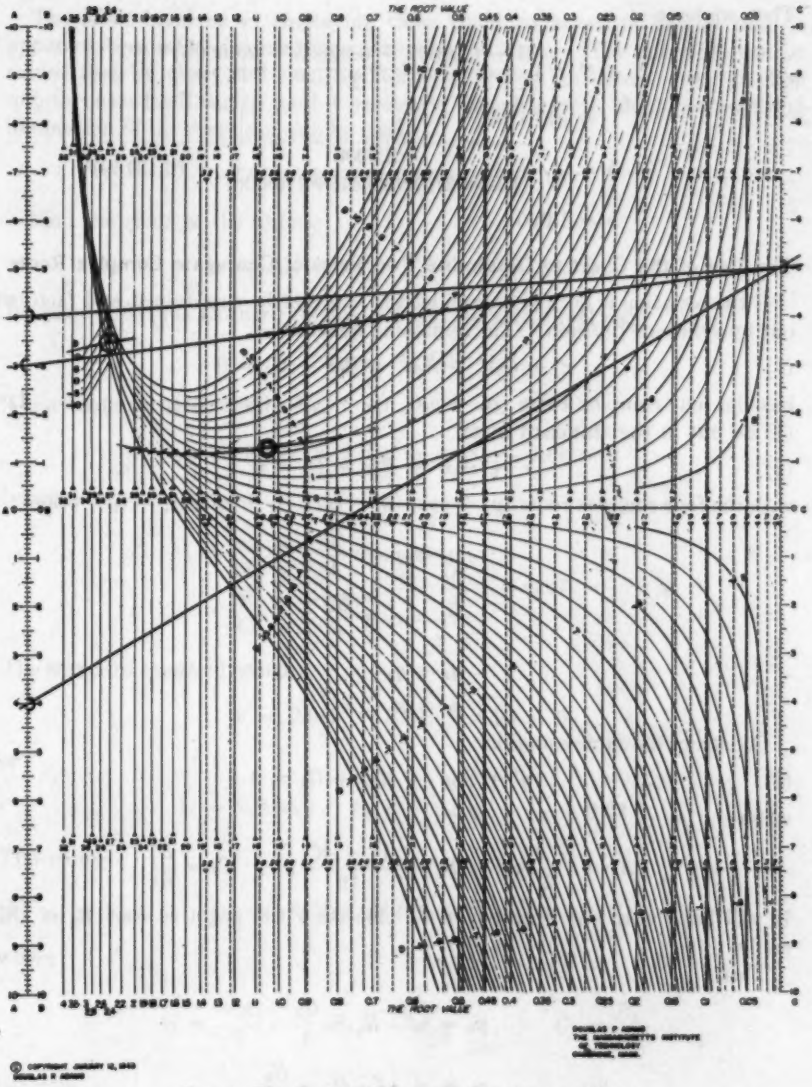

FIG. 8. The Quintic Hypernom for the equation

$$x^5 + A \cdot x^3 + B \cdot x^2 + C \cdot x + D = 0$$

Horner's method now yields the roots:

1.21602
0.92665
−2.40606

Then we have

$$a = -\left\{\frac{0 + 1.20602 + 0.92665 - 2.40606}{2}\right\}$$

$$= (0.13169)$$

$$b = \sqrt{\frac{+2.9000}{(1.21602)(0.92665)(+2.40606)} - (0.13169)^2}$$

$$b = 1.026$$

The Use of the Chart to Determine Two Pairs of Conjugate Complex Roots

If the equation is of fifth degree, let the real root be ascertained and factored out by synthetic division. The remaining quartic

(1) $$x^4 + G_0x^3 + A_0x^2 + B_0x + C_0 = 0$$

has the four complex roots $(p_0 + q_0i)$, $(p_0 - q_0i)$, $(r_0 + s_0i)$, $(r_0 - s_0i)$. Let (1) be put in the standard form

(2) $$x^4 + A_1x^2 + B_1x + C_1 = 0$$

with the four complex roots $(p_1 + q_1i)$, $(p_1 - q_1i)$, $(r_1 + s_1i)$, $(r_1 - s_1i)$, where

$$p_1 = p_0 + \frac{G_0}{4}$$

$$r_1 = r_0 + \frac{G_0}{4}$$

$$q_1 = q_0$$

$$s_1 = s_0$$

Construct the cubic equation

(3) $$y^3 + A_2y^2 + B_2y + C_2 = 0$$

where

$$A_2 = \frac{-A_1}{2}, \qquad B_2 = \frac{A_1^2 - 4C_1}{16}, \qquad C_2 = \frac{B_1^2}{64}$$

It will have only one negative root. Ascertain the negative root R_y of this cubic.

Then

$$p_0 = \sqrt{-R_y} - \frac{G_0}{4}$$

$$r_0 = -\sqrt{-R_y} - \frac{G_0}{4}$$

$$q_0 = \sqrt{\frac{A_1}{2} - R_y + \frac{B_1}{4\sqrt{-R_y}}}$$

$$s_0 = \sqrt{\frac{A_1}{2} - R_y - \frac{B_1}{4\sqrt{-R_y}}}$$

The labor involved in employing these latter equations is smaller than is apparent because of frequent repetitions in the terms. The cubic (3) can be solved first by chart and then by Horner's method for precise results. The quintic chart itself can be used if necessary, though a cubic chart not requiring diminution of the roots can also be used.

Illustrative Problem 6

Find the roots of the quintic

$$x^5 + 5x^4 + 7x^3 - 3x^2 - 4x + 3 = 0$$

By Figure (5) plus Horner's method we find the single real root

$$x = -1.1186$$

The remaining quartic is

$$x^4 + 3.88x^3 + 2.66x^2 - 5.98x + 2.69 = 0$$

hence

$$\begin{aligned} G_0 &= 3.88 \\ A_0 &= 2.66 \\ B_0 &= -5.98 \\ C_0 &= 2.69 \end{aligned} \qquad -\frac{G_0}{4} = -0.97$$

The standard quartic becomes

$$x^4 - 2.98x^2 - 3.84x + 8.34 = 0$$

or

$$A_1 = -2.98, \qquad B_1 = -3.84, \qquad C_1 = 8.34$$

The cubic

$$y^3 + A_2y^2 + B_2y + C_2 = 0 \tag{3}$$

where

$$A_2 = \frac{-A_1}{2}, \qquad B_2 = \frac{A_1^2 - 4C_1}{16}, \qquad C_2 = \frac{B_1^2}{64}$$

becomes

$$y^3 + 1.49y^2 - 1.53y + .231 = 0 \tag{3}$$

The negative root of this equation is

$$R_y = -2.224$$

Then

$$p_0 = \sqrt{-R_y} - \frac{G_0}{4} = \sqrt{2.224} - 0.97 = 0.521$$

$$r_0 = \sqrt{-R_y} - \frac{G_0}{4} = -\sqrt{2.224} - 0.97 = -2.461$$

$$q_0 = \sqrt{\frac{A_1}{2} - R_y + \frac{B_1}{4\sqrt{-R_y}}} = \sqrt{-1.49 + 2.224 + \frac{-3.84}{4\sqrt{2.224}}} = 0.298$$

$$s_0 = \sqrt{\frac{A_1}{2} - R_y - \frac{B_1}{4\sqrt{-R_y}}} = \sqrt{-1.49 + 2.224 - \frac{-3.84}{4\sqrt{2.224}}} = 1.173$$

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

POLYADIC PRODUCTS

By I. M. Hostetter

This is the first of several papers dealing with the extension of the methods used by the writer in the treatment of vector spaces[1] to the study of polyadics. Naess[2] defined a vector product for n-space by means of a symbolic determinant which he called a space complement. Hitchcock and Rice[3] extended Gibbs' multiple cross products to polyadics in general by generalizing Naess' complement.

The present paper attempts to avoid some of the difficulties of notation of these writers in a more direct employment of the Gibbs' notation to the definition of a variety of operational functions of polyadics, which for want of a better term will be called products. Certain sets of these operational functions applied to a polyadic $\mathbf{A}$ are shown to characterize the ranks of the various matrices associated with $\mathbf{A}$. With their aid sets of scalar invariants of $\mathbf{A}$ are also obtained which are shown to exist as the coefficients in the expansion of certain sets of characteristic λ determinants.

1. **Definitions, etc.** Let $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \cdots$ be a set of elements of a vector space $\mathfrak{P}^{(1)}$ over the field F of real numbers. The dimensionality of this space will be designated by n, which is supposed finite. The "open" or "indeterminate" product formed by the juxtaposition of two or more vectors of $\mathfrak{P}^{(1)}$, $\mathbf{b}_1\mathbf{b}_2 \cdots \mathbf{b}_p$, is known as a *polyad*. The number of vector factors in a polyad is termed the *class* of the polyad, the dimensionality n of the vector factors the *order* of the polyad.

A polyadic $\mathbf{A}$ is the indicated sum of one or more polyads of the same class,

$$\mathbf{A} = \sum_{i=1}^{k} \mathbf{b}_{1i}\mathbf{b}_{2i} \cdots \mathbf{b}_{pi}, \qquad (k \geqq 1). \tag{1.1}$$

The class and order of a polyadic are the class and order of its constituent polyads. A polyadic of class one is thus a vector and of class zero a scalar. When it is desired to emphasize the class of a polyadic it will be indicated by a subscript or superscript containing the class in parentheses; thus $\mathbf{A}_{(c)}$ or $\mathbf{A}^{(c)}$. Such an index will generally be placed or removed without comment.

It is assumed that for any polyadics $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and any α and β in F that the following properties hold:

$$\begin{gathered}\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}, \quad \mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}, \quad (\alpha\beta)\mathbf{A} = \alpha(\beta\mathbf{A}) = \mathbf{A}(\alpha\beta) \\ \alpha(\mathbf{A} + \mathbf{B}) = \alpha\mathbf{A} + \alpha\mathbf{B}, \quad (\alpha + \beta)\mathbf{A} = \alpha\mathbf{A} + \beta\mathbf{A}.\end{gathered} \tag{1.2}$$

[1] *An extension of Gibbs' Vector Analysis to* n*-space*, Journal of Mathematics and Physics, Vol. 15 (1936), pp. 195–204.

[2] Naess, Almar, *On a special polyadic . . . which can be regarded as a generalization of the vector product*, Videnskaposelskapets Skrifter Mat.-naturv., Klasse, 1922, No. 13.

[3] Hitchcock, F. L., Rice, L. H., *The multiple complement of one or more polyadics*, Jour. Math. and Physics, Vol. 4 (1925), pp. 179–187.

We will call the set of all polyadics of class p which satisfy the conditions stated above a polyadic space $\mathfrak{P}^{(p)}(F)$. This terminology may be objected to by some on the ground that such spaces are included under the general classification of vector spaces. Polyadic spaces, however, possess many properties not commonly attributed to vector spaces, and the writer finds it convenient to consider vector spaces as subspace of all polyadic space, namely the space $\mathfrak{P}^{(1)}$. The terminology used in the study of vector spaces will be applied to polyadic spaces without definition whenever the meaning is clear from analogy.

As is well known, every polyadic of class p can be expressed in the form

$$\mathbf{A} = A^{\alpha_1 \cdots \alpha_p} \mathbf{a}_{\alpha_1} \cdots \mathbf{a}_{\alpha_p}, \qquad (\alpha_k = 1, \cdots, n), \tag{1.3}$$

where the quantities are summed on the repeated indices and the $\mathbf{a}_\alpha$ constitute a basis for the vector space $\mathfrak{P}^{(1)}$. The n^p coefficients $A^{\alpha_1 \cdots \alpha_p}$ constitute the components of $\mathbf{A}$ with respect to the basis vectors $\mathbf{a}_1, \cdots, \mathbf{a}_n$, and will often be used to designate the polyadic when no ambiguity can result.

The components of a polyadic are not to be confused with the components of a tensor. Only those polyadics whose components are capable of transformation under the particular transformations known as covariant or contravariant are tensors. A polyadic, however, may be subjected to an arbitrary transformation on any index or group of indices as desired.

A *null* polyadic is one which when reduced to the form (1.3) has all its components zero. Two polyadics are equal if, and only if, corresponding components are equal when both polyadics are expressed in terms of the same basis vectors.

If every polyad of a polyadic $\mathbf{A}$ has its vector factors permuted in the same manner the resulting polyadic will be called an isomer of $\mathbf{A}$, indicated by $\mathbf{A}'$. If any two or more isomers are equal the polyadic will be said to be *symmetric* on the indices involved. Bold face type will always be used to indicate vectors or polyadics.

2. **Linear dependence of polyadics.** If the m polyadics $\mathbf{P}_i$ are linearly dependent there exist m constants μ^i in F, not all zero, such that

$$\mu^i \mathbf{P}_i = \mathbf{O},$$

where $\mathbf{O}$ indicates a null polyadic. That is, if each $\mathbf{P}_i$ is expressed in terms of the same basis vectors we have the n^p scalar equations

$$\mu^i P_i^{\alpha_1 \cdots \alpha_p} = 0.$$

The quantities $P_i^{\alpha_1 \cdots \alpha_p}$ may be displayed as a two-way matrix whose rows are the files on the index i of the $(p+1)$-way matrix $\| P_i^{\alpha_1 \cdots \alpha_p} \|$. We represent this matrix by $\| P_i^u \|$ where $u = \alpha_1 \cdots \alpha_p$ has the range $1, \cdots, n^p$ and i the range $1, \cdots, m$. u is called a *partition* of the indices α_k. Thus for $m = 3$, $n = 2$,

$$\| P_i^u \| = \begin{Vmatrix} P_1^{11} P_2^{11} P_3^{11} \\ P_1^{12} P_2^{12} P_3^{12} \\ P_1^{21} P_2^{21} P_3^{21} \\ P_1^{22} P_2^{22} P_3^{22} \end{Vmatrix} \tag{2.1}$$

It follows immediately that

THEOREM 1. *The necessary and sufficient condition that m polyadics $\mathbf{P}_i$ of class p be linearly dependent is that all the m-rowed minors of the matrix $\| P_i^u \|$, $(u = \alpha_1 \cdots \alpha_p ; i = 1, \cdots, m)$ vanish.*

As an immediate consequence we have

THEOREM 2. *Any n^p independent polyadics of class p form a basis for a polyadic space $\mathfrak{P}^{(p)}$.*

Let the indices of $\mathbf{A}$ be grouped in any way into t sets of $c_1, \cdots, c_t$ indices each. This can evidently be done in a great many ways since the indices of each set need not be consecutive. For convenience of notation, we will select that isomer of $\mathbf{A}$ which so arranges the vector files of $\mathbf{A}$ that the indices of each set will be consecutive. By considering the properties of this polyadic there will be no loss in generality, since the original polyadic can always be obtained, regardless of the attained form, by changing back to the original order of indices and interpreting any properties in terms of the rearranged indices. We consider then the polyadic

$$\mathbf{A}' = A^{\alpha_{11}\cdots\alpha_{1c},\cdots,\alpha_{t1}\cdots\alpha_{td}}(\mathbf{a}_{\alpha_{11}} \cdots \mathbf{a}_{\alpha_{1c}}) \cdots (\mathbf{a}_{\alpha_{t1}} \cdots \mathbf{a}_{\alpha_{td}}), \quad (c = c_1; d = c_t) \tag{2.2}$$

In this equation we make the substitutions

$$\mathbf{a}_{\alpha_{k1}} \cdots \mathbf{a}_{\alpha_{kh}} = \mathbf{B}^{(c_k)}_{\alpha_{k1}\cdots\alpha_{kh}}, \qquad (h = c_k)$$

and take the partitions $u_k = \alpha_{k1} \cdots \alpha_{kh}$ on the indices, obtaining

$$\mathbf{A}' = A^{u_1\cdots u_t}\mathbf{B}^{(c_1)}_{u_1} \cdots \mathbf{B}^{(c_t)}_{u_t}.$$

Since each of the polyads $\mathbf{B}^{(c_k)}_{u_k}$ may be expressed in terms of n^p arbitrarily selected basis polyadics $\mathbf{P}_i^{(c_i)}$ it follows that

THEOREM 3. *An isomer of any polyadic $\mathbf{A}$ of class p may be expressed in the form*

$$\mathbf{A}' = p^{u_1\cdots u_t}\mathbf{P}^{(c_1)}_{u_1} \cdots \mathbf{P}^{(c_t)}_{u_t}, \qquad (u_j = 1, \cdots, n^{c_j}), \tag{2.3}$$

where each set of polyadics $\mathbf{P}^{(c_j)}_{u_j}$ is an arbitrarily selected set of independent polyadics of class c_j.

By placing

$$p^{u_1\cdots u_t}\mathbf{P}^{(c_q)}_{u_q} = \mathbf{P}^{u_1\cdots u_r u_s \cdots u_t}_{(c_q)}, \qquad (r = q - 1; s = q + 1),$$

in (2.3) we obtain the useful corollary

COROLLARY. *An isomer of any polyadic $\mathbf{A}$ can be developed on the multipartite indices $u_1, \cdots, u_t$ as*

$$\mathbf{A}' = \mathbf{P}^{(c_1)}_{u_1} \cdots \mathbf{P}^{(c_r)}_{u_r}\mathbf{P}^{u_1\cdots u_r u_s\cdots u_t}_{(c_q)}\mathbf{P}^{(c_s)}_{u_s} \cdots \mathbf{P}^{(c_t)}_{u_t}, \tag{2.4}$$
$$(u_j = 1, \cdots, n^{c_j}; r = q - 1; s = q + 1),$$

where each of the $t-1$ sets of polyadics $\mathbf{P}_{u_j}^{(e_j)}$ is a basis for a polyadic space $\mathfrak{P}^{(e_j)}$, and the polyadics $\mathbf{P}_{(e_q)}^{u_1\cdots u_r u_s\cdots u_t}$ are elements of a $\mathfrak{P}^{(e_q)}$.

3. **The multiple inner product. A metric for polyadics.** Let

$$\mathbf{B} = \mathbf{b}_{1,i}\cdots\mathbf{b}_{p,i}, \qquad \mathbf{C} = \mathbf{c}_{1,j}\cdots\mathbf{c}_{q,j}, \qquad (i,j = 1,\cdots,h),$$

be two real polyadics of classes p and q, respectively, $(p \geqq q)$. The multiple inner product $\mathbf{B}:\mathbf{C}$ of $\mathbf{B}$ and $\mathbf{C}$ is defined by the relation

$$(3.1)\qquad \mathbf{B}:\mathbf{C} = (\mathbf{b}_{1,i}\cdots\mathbf{b}_{p,i}):(\mathbf{c}_{1,j}\cdots\mathbf{c}_{q,j}) = \mathbf{b}_{1,i}\cdot\mathbf{c}_{1,j}\cdots\mathbf{b}_{q,i}\cdot\mathbf{c}_{q,j}\mathbf{b}_{q+1,i}\cdots\mathbf{b}_{p,i}.$$

Since the inner products $\mathbf{b}_{k,i}\cdot\mathbf{c}_{k,j}$ are commutative, it follows that the multiple inner product is commutative.

$$\mathbf{B}:\mathbf{C} = \mathbf{C}:\mathbf{B}$$

If we set

$$\mathbf{b}_{ki} = \beta_{k,i}^{\alpha}\,\mathbf{a}_\alpha = \beta_{k,i\alpha}\mathbf{a}^\alpha, \qquad \mathbf{c}_{k,j} = \gamma_{k,j}^{\beta}\mathbf{a}_\beta = \gamma_{k,j\beta}\mathbf{a}^\beta$$

where the basis vectors $\mathbf{a}^\alpha$ and $\mathbf{a}_\alpha$ are elements of reciprocal[4] sets, for which $\mathbf{a}^\alpha\cdot\mathbf{a}_\beta = \delta_\beta^\alpha$, and also let

$$\beta_{1,i}^{\alpha_1}\cdots\beta_{p,i}^{\alpha_p} = B^{\alpha_1\cdots\alpha_p}, \qquad \beta_{1,i\alpha_1}\cdots\beta_{p,i\alpha_p} = B_{\alpha_1\cdots\alpha_p},$$

$$\gamma_{1,j}^{\beta_1}\cdots\gamma_{q,j}^{\beta_p} = C^{\beta_1\cdots\beta_q}, \qquad \gamma_{1,j\beta_1}\cdots\gamma_{p,j\beta_q} = C_{\beta_1\cdots\beta_q},$$

the multiple inner product may be found in the forms

$$\begin{aligned}\mathbf{B}:\mathbf{C} &= B^{\alpha_1\cdots\alpha_p}C^{\beta_1\cdots\beta_q}\mathbf{a}_{\alpha_1}\cdot\mathbf{a}_{\beta_1}\cdots\mathbf{a}_{\alpha_q}\cdot\mathbf{a}_{\beta_q}\mathbf{a}_{\alpha_{q+1}}\cdots\mathbf{a}_{\alpha_p},\\ &= B_{\alpha_1\cdots\alpha_p}C_{\beta_1\cdots\beta_q}\mathbf{a}^{\alpha_1}\cdot\mathbf{a}^{\beta_1}\cdots\mathbf{a}^{\alpha_q}\cdot\mathbf{a}^{\beta_q}\mathbf{a}^{\alpha_{q+1}}\cdots\mathbf{a}^{\alpha_p},\\ &= B_{\alpha_1\cdots\alpha_p}C^{\alpha_1\cdots\alpha_q}\mathbf{a}^{\alpha_{q+1}}\cdots\mathbf{a}^{\alpha_p} = B^{\alpha_1\cdots\alpha_p}C_{\alpha_1\cdots\alpha_q}\mathbf{a}_{\alpha_{q+1}}\cdots\mathbf{a}_{\alpha_p}.\end{aligned}$$

Again placing

$$(3.2)\qquad \mathbf{a}_{\alpha_1}\cdot\mathbf{a}_{\beta_1}\cdots\mathbf{a}_{\alpha_q}\cdot\mathbf{a}_{\beta_q} = G_{\alpha_1\cdots\alpha_q,\beta_1\cdots\beta_q}, \qquad \mathbf{a}^{\alpha_1}\cdot\mathbf{a}^{\beta_1}\cdots\mathbf{a}^{\alpha_q}\cdot\mathbf{a}^{\beta_q} = G^{\alpha_1\cdots\alpha_q,\beta_1\cdots\beta_1},$$

and making the partitions $i = \alpha_1\cdots\alpha_q$, $j = \beta_1\cdots\beta_q$ the result is

$$\begin{aligned}\mathbf{B}:\mathbf{C} &= G_{ij}B^{i,\alpha_{q+1}\cdots\alpha_p}C^j\mathbf{a}_{\alpha_{q+1}}\cdots\mathbf{a}_{\alpha_p},\\ &= G^{ij}B_{i,\alpha_{q+1}\cdots\alpha_p}C_j\mathbf{a}^{\alpha_{q+1}}\cdots\mathbf{a}^{\alpha_p},\\ &= B_{i,\alpha_{q+1}\cdots\alpha_p}C^i\mathbf{a}^{\alpha_{q+1}}\cdots\mathbf{a}^{\alpha_p} = B^{i,\alpha_{q+1}\cdots\alpha_p}C_i\mathbf{a}_{\alpha_{q+1}}\cdots\mathbf{a}_{\alpha_p}.\end{aligned}\qquad (3.3)$$

When several polyadics are involved in a multiple inner product, the indeterminate products are considered as taken first and the inner product applied to the result as in (3.1). Thus

$$(3.4)\qquad \mathbf{A}:\mathbf{BC} = \mathbf{A}:(\mathbf{BC}), \qquad \mathbf{AB}:\mathbf{C} = (\mathbf{AB}):\mathbf{C},$$

[4] Cf. Hostetter, loc. cit. p. 195. It is to be noticed that the order of application of the multiple inner product in this paper follows that of Gibbs, which is the exact opposite to that of the present paper.

where the quantities in parentheses are to be considered as single polyadics and the process of taking the inner product of corresponding vectors is carried out, starting from the left, until all the vector fcators of one side or the other are exhausted. If it is desired to limit or emphasize the number of factors to which the multiple product is applied, the same will be indicated by a subscript on the double dot. Thus corresponding to (3.1)

$$\mathbf{B}:_r\mathbf{C} = \mathbf{b}_{1,i}\cdot\mathbf{c}_{1,j}\cdots\mathbf{b}_{r,i}\cdot\mathbf{c}_{r,j}\mathbf{b}_{r+1,i}\cdots\mathbf{b}_{p,i}\mathbf{c}_{r+1,j}\cdots\mathbf{c}_{q,j}. \tag{3.5}$$

It is obvious that such limited inner products are not commutative.

DEFINITION. *The norm of a polyadic* $\mathbf{A}$ *is the invariant* $|\mathbf{A}|$ *defined by*

$$|\mathbf{A}| = \sqrt{\mathbf{A}:\bar{\mathbf{A}}}. \tag{3.6}$$

With the aid of (3.3) we see that

$$|\mathbf{A}|^2 = G_{ij}A^iA^j = G^{ij}A_iA_j = A^iA_i.$$

These forms are positive definite since the vector factors of each polyad of $\mathbf{A}$ are taken over the field of real numbers. Hence $|\mathbf{A}|$ is always real.

If $|\mathbf{A}| = 1$ we will call $\mathbf{A}$ a unit polyadic. The norm of any polyadic has the following properties corresponding to those of length in euclidean vector space:

$$\begin{aligned} &|k\mathbf{A}| = |k|\,|\mathbf{A}|, && |\mathbf{A}:\mathbf{B}| \leqq |\mathbf{A}|\,|\mathbf{B}|, \\ &|\mathbf{A}| > 0 \quad \text{for} \quad \mathbf{A} \neq 0, && |\mathbf{A}+\mathbf{B}| \leqq |\mathbf{A}| + |\mathbf{B}|, \end{aligned} \tag{3.7}$$

where k is real and $\mathbf{A}$ and $\mathbf{B}$ are both of the same class.

DEFINITION. *Any two polyadics* $\mathbf{A}$ *and* $\mathbf{B}$ *such that for some isomers* $\mathbf{A}'$ *and* $\mathbf{B}'$ *the relation*

$$\mathbf{A}':_r\mathbf{B}' = 0$$

is satisfied will be said to be orthogonal on the r *indices involved.*

Any set of mutually orthogonal unit polyadics of the same class will be denoted by $\mathbf{E}_i$. By definition then

$$\mathbf{E}_i:\mathbf{E}_j = \delta_{ij}. \tag{3.8}$$

If the class is one the $\mathbf{E}_i$ will be vectors and will be denoted by $\mathbf{e}_i$. A set $\mathbf{E}_i$ satisfying (3.8) is independent for, if there exist constants λ^i such that $\lambda^i\mathbf{E}_i = 0$, it follows from (3.8) that

$$\lambda^i\mathbf{E}_i:\mathbf{E}_j = \lambda^j = 0.$$

The simplest set of orthogonal unit polyadics forming a basis for a $\mathfrak{P}^{(p)}$ is formed by setting the polyads $\mathbf{e}_{\alpha_1}\cdots\mathbf{e}_{\alpha_p}$, $(\alpha_i = 1, \cdots, n)$, equal to $\mathbf{E}_1$, $\mathbf{E}_2$, etc. Thus for $n = 2, p = 2$

$$\mathbf{E}_1 = \mathbf{e}_1\mathbf{e}_1, \qquad \mathbf{E}_2 = \mathbf{e}_1\mathbf{e}_2, \qquad \mathbf{E}_3 = \mathbf{e}_2\mathbf{e}_1, \qquad \mathbf{E}_4 = \mathbf{e}_2\mathbf{e}_2.$$

It is obvious that Theorem 3 and its corollary are valid when the basis polyadics in each expansion are interpreted as orthogonal unit polyadics.

4. **The multiplex outer product.** We have need of the following:

LEMMA. *There exist an infinite number of sets of $\nu - m$ independent polyadics $\mathbf{B}_\lambda$ of any subspace $\mathfrak{P}_\nu^{(p)}$ of $\mathfrak{P}^{(p)}$ which are orthogonal to any set of m independent polyadics $\mathbf{A}_\gamma$ of $\mathfrak{P}_\nu^{(p)}$.*

PROOF. Let $\mathbf{P}_i$, $(i = 1, \cdots, \nu)$, be a basis for $\mathfrak{P}_\nu^{(p)}$; then we can write

$$\mathbf{A}_\gamma = \alpha_\gamma^i \mathbf{P}_i, \qquad \mathbf{B}_\lambda = \beta_\lambda^j \mathbf{P}_j,$$

$$(\gamma = 1, \cdots, m; \quad \lambda = 1, \cdots, \nu - m; \quad i, j = 1, \cdots, \nu).$$

The problem is than that of finding $\nu - m$ independent sets of ν constants $\beta_\lambda^1, \cdots, \beta_\lambda^\nu$ such that

$$\mathbf{A}_\gamma : \mathbf{B}_\lambda = \alpha_\gamma^i \beta_\lambda^j \mathbf{P}_i : \mathbf{P}_j = \mu_\gamma \beta_\lambda^j = 0.$$

The lemma follows from the well known theorem of algebra.[5]

Let $\mathbf{B}_\lambda$, $(\lambda = 3, \cdots, \nu)$, be any $\nu - 2$ such polyadics, elements of a $\mathfrak{P}_\nu^{(p)}$, orthogonal to $\mathbf{A}_1$ and $\mathbf{A}_2$. We introduce a function of the indeterminate product of $\mathbf{A}_1$ and $\mathbf{A}_2$, written $\mathbf{A}_1 * \mathbf{A}_2$, which we will call the *multiplex outer product*, or *star product* of these polyadics. We define this product over the subspace $\mathfrak{P}_\nu^{(p)}$ of the polyadics $\mathbf{A}_1, \mathbf{A}_2, \mathbf{B}_3, \cdots, \mathbf{B}_\nu$ as a polyadic $\beta\mathbf{M}$ given by

$$(4.1) \qquad {}^\nu(\mathbf{A}_1 * \mathbf{A}_2) = \beta \epsilon^{12 i_3 \cdots i_\nu} \mathbf{B}_{i_3} \cdots \mathbf{B}_{i_\nu} = \beta \mathbf{M}, \qquad (2 < \nu \leqq n^p; \qquad i_k = 3, \cdots, \nu),$$

where

$$(4.2) \qquad \beta = \left\{ \frac{(\nu - 2)!}{\mathbf{M} : \mathbf{M}} \begin{vmatrix} \mathbf{A}_1 : \mathbf{A}_1 & \mathbf{A}_1 : \mathbf{A}_2 \\ \mathbf{A}_2 : \mathbf{A}_1 & \mathbf{A}_2 : \mathbf{A}_2 \end{vmatrix} \right\}^{\frac{1}{2}}$$

and $\epsilon^{12 i_3 \cdots i_\nu}$ has the usual meaning of this symbol in the tensor analysis. ${}^\nu(\mathbf{A}_1 * \mathbf{A}_2)$ is thus a polyadic of class $p^{\nu-2}$. If $\nu = n^p$, that is, if the star product is taken over polyadic space $\mathfrak{P}^{(p)}$, the pre-superscript n^p will be omitted; thus,

$${}^{n^p}(\mathbf{A}_1 * \mathbf{A}_2) = \mathbf{A}_1 * \mathbf{A}_2 .$$

Taking the multiple inner product of both members of (4.1) we have, with the aid of (4.2) and (3.6),

$${}^\nu(\mathbf{A}_1 * \mathbf{A}_2) : {}^\nu(\mathbf{A}_1 * \mathbf{A}_2) = \beta^2 \mathbf{M} : \mathbf{M} = (\nu - 2)! \begin{vmatrix} \mathbf{A}_1 : \mathbf{A}_1 & \mathbf{A}_1 : \mathbf{A}_2 \\ \mathbf{A}_2 : \mathbf{A}_1 & \mathbf{A}_2 : \mathbf{A}_2 \end{vmatrix}.$$

That is,

$$| {}^\nu(\mathbf{A}_1 * \mathbf{A}_2) |^2 = (\nu - 2)! \begin{vmatrix} \mathbf{A}_1 : \mathbf{A}_1 & \mathbf{A}_1 : \mathbf{A}_2 \\ \mathbf{A}_2 : \mathbf{A}_1 & \mathbf{A}_2 : \mathbf{A}_2 \end{vmatrix}.$$

THEOREM 4. *The star product of two polyadics is unique.*

PROOF. Since the theorem is unquestionably true if $\mathbf{A}_1$ and $\mathbf{A}_2$ are linearly related, we have to consider only the case where these polyadics are independent. Let

$$\beta_1 \mathbf{M} = \frac{\lambda}{(\mathbf{M} : \mathbf{M})^{\frac{1}{2}}} \epsilon^{12 i_3 \cdots i_\nu} \mathbf{B}_{i_3} \cdots \mathbf{B}_{i_\nu},$$

[5] Bocher, Maxime, *Introduction to higher algebra*, p. 50.

and

$$\beta_2\mathbf{N} = \frac{\lambda}{(\mathbf{N}:\mathbf{N})^{\frac{1}{2}}}\epsilon^{12j_3\cdots j_\nu}\mathbf{C}_{j_3}\cdots\mathbf{C}_{j_\nu},$$

where $\lambda = |{}'(\mathbf{A}_1{*}\mathbf{A}_2)|$ and i_k, $j_k = 3, \cdots, \nu$, be two representations of the product ${}'(\mathbf{A}_1{*}\mathbf{A}_2)$. Since the $\mathbf{C}_j$ are independent they form a basis for the subspace $\mathfrak{P}_{\nu-2}^{(p)}$, which is the orthogonal complement to the $\mathfrak{P}_2^{(p)}$ determined by $\mathbf{A}_1$ and $\mathbf{A}_2$. Since the $\mathbf{B}_i$ are elements of $\mathfrak{P}_{\nu-2}^{(p)}$ we can therefore write

$$\mathbf{B}_i = \gamma_i^k\mathbf{C}_k,$$

and

$$\beta_1\mathbf{M} = \frac{\lambda\epsilon^{12i_3\cdots i_\nu}\gamma_{i_3}^{k_3}\cdots\gamma_{i_\nu}^{k_\nu}\mathbf{C}_{k_3}\cdots\mathbf{C}_{k_\nu}}{(\epsilon^{12j_3\cdots j_\nu}\epsilon^{12k_3\cdots k_\nu}\mathbf{B}_{j_3}:\mathbf{B}_{k_3}\cdots\mathbf{B}_{j_\nu}:\mathbf{B}_{k_\nu})^{\frac{1}{2}}},$$

or

$$\beta_1\mathbf{M} = \frac{\lambda\,|\,\gamma\,|\,\epsilon^{12k_3\cdots k_\nu}\mathbf{C}_{k_3}\cdots\mathbf{C}_{k_\nu}}{[(\nu-2)!\,|\,\mathbf{B}_i:\mathbf{B}_j\,|]^{\frac{1}{2}}} = \frac{\lambda\epsilon^{12k_3\cdots k_\nu}\mathbf{C}_{k_3}\cdots\mathbf{C}_{k_\nu}}{[(\nu-2)!\,|\,\mathbf{C}_i:\mathbf{C}_j\,|]^{\frac{1}{2}}}.$$

Since from the equivalent of (4.1)

$$\mathbf{N}:\mathbf{N} = |\,\mathbf{C}_i:\mathbf{C}_j\,|(\nu-2)!,$$

it follows that

$$\beta_1\mathbf{M} = \beta_2\mathbf{N}.$$

Q.E.D.

For an orthogonal set of unit polyadics $\mathbf{E}_i$ we have for the multiplex outer product of any two

$${}'(\mathbf{E}_{i_1}{*}\mathbf{E}_{i_2}) = \beta\epsilon^{i_1\cdots i_\nu}\mathbf{E}_{i_3}\cdots\mathbf{E}_{i_\nu}, \qquad (i_k = 1, \cdots, \nu).$$

But

$$\beta^2 = \frac{(\nu-2)!\,|\,\mathbf{E}_h:\mathbf{E}_k\,|}{\epsilon^{i_1i_2i_3\cdots i_\nu}\epsilon^{i_1i_2j_3\cdots j_\nu}\mathbf{E}_{i_3}:\mathbf{E}_{j_3}\cdots\mathbf{E}_{i_\nu}:\mathbf{E}_{j_\nu}} = \frac{(\nu-2)!}{\epsilon^{i_1i_2i_3\cdots i_\nu}\epsilon^{i_1i_2i_3\cdots i_\nu}} = 1.$$

Hence

$$(4.3) \qquad {}'(\mathbf{E}_{i_1}{*}\mathbf{E}_{i_2}) = \epsilon^{i_1\cdots i_\nu}\mathbf{E}_{i_3}\cdots\mathbf{E}_{i_\nu}, \qquad (i_k = 1, \cdots, \nu).$$

It is readily seen that the star product reduces to the cross product of Gibbs for $p = 1$, $\nu = n = 3$.

If $\mathbf{A}^{(c+m)}$ is a polyadic of class $c + m$, according to the corollary to Theorem 3 we can write

$$\mathbf{A} = \mathbf{M}_{(m)}^i\mathbf{C}_i^{(c)}, \qquad (i = 1, \cdots, \nu).$$

If $\mathbf{B}^{(c)}$ is another polyadic of class c, then we interpret the star product of $\mathbf{A}$ and $\mathbf{B}$ to mean

$$(4.4) \qquad {}'(\mathbf{A}^{(c+m)}{*}\mathbf{B}^{(c)}) = \mathbf{M}_{(m)}^i{}'(\mathbf{C}_i^{(c)}{*}\mathbf{B}^{(c)}).$$

Star products satisfy the following laws for all real scalars ξ and η and all polyadics $\mathbf{A}$ in $\mathfrak{P}_\nu^{(p)}$.

$$
\begin{aligned}
&(\xi + \eta)\,{}^{\nu}(\mathbf{A}_1{*}\mathbf{A}_2) = \xi\,{}^{\nu}(\mathbf{A}_1{*}\mathbf{A}_2) + \eta\,{}^{\nu}(\mathbf{A}_1{*}\mathbf{A}_2), && \text{(i)}\\
&{}^{\nu}[(\xi\mathbf{A}_1){*}(\eta\mathbf{A}_2)] = \xi\eta\,{}^{\nu}(\mathbf{A}_1{*}\mathbf{A}_2), && \text{(ii)}\\
&{}^{\nu}[(\mathbf{A}_1 + \mathbf{A}_2){*}\mathbf{A}_3] = {}^{\nu}(\mathbf{A}_1{*}\mathbf{A}_3) + {}^{\nu}(\mathbf{A}_2{*}\mathbf{A}_3), && \text{(iii)}\\
&{}^{\nu}(\mathbf{A}_1{*}\mathbf{A}_2) = -{}^{\nu}(\mathbf{A}_2{*}\mathbf{A}_1), \quad \text{(iv)}; \qquad {}^{\nu}(\mathbf{A}_1{*}\mathbf{A}_1) \geqq 0. && \text{(v)}
\end{aligned}
\tag{4.5}
$$

(i) is a consequence of the last of relations (1.2) since the multiplex outer product of two polyadics is itself a polyadic. (ii) follows immediately from the definition of the star product. For if in (4.1) the polyadics $\mathbf{A}_1$ and $\mathbf{A}_2$ are multiplied by the scalars ξ and η, respectively, the resulting β of the right member would be the β of (4.1) multiplied by $\xi\eta$ as is seen by consideration of (4.2).

To show that (iii) holds, we will first show that

$$
{}^{\nu}(\mathbf{E}_1 + \mathbf{E}_2){*}\mathbf{E}_3] = {}^{\nu}(\mathbf{E}_1{*}\mathbf{E}_3) + {}^{\nu}(\mathbf{E}_2{*}\mathbf{E}_3). \tag{4.6}
$$

From (4.3) it follows that

$$
{}^{\nu}(\mathbf{E}_1{*}\mathbf{E}_3) = \epsilon^{13i_3\cdots i_\nu}\mathbf{E}_{i_3}\cdots\mathbf{E}_{i_\nu}, \qquad {}^{\nu}(\mathbf{E}_2{*}\mathbf{E}_3) = \epsilon^{23j_3\cdots j_\nu}\mathbf{E}_{j_3}\cdots\mathbf{E}_{j_\nu}.
$$

Adding we obtain

$$
{}^{\nu}(\mathbf{E}_1{*}\mathbf{E}_3) + {}^{\nu}(\mathbf{E}_2{*}\mathbf{E}_3) = \sqrt{2}\,\epsilon^{123k_3\cdots k_\nu}\mathbf{E}_{k_3}\cdots\mathbf{E}_{k_\nu}, \qquad (k_j = 0, 1, \cdots, \nu), \tag{4.7}
$$

where

$$
\mathbf{E}_0 = \frac{\mathbf{E}_1 - \mathbf{E}_2}{\sqrt{2}}.
$$

$\mathbf{E}_0$ is a unit polyadic satisfying the conditions

$$
(\mathbf{E}_1 + \mathbf{E}_2):\mathbf{E}_0 = 0, \qquad \mathbf{E}_0:\mathbf{E}_j = \delta_{0j} \qquad (j = 0, 3, 4, \cdots, \nu),
$$

and, as a consequence, we can write

$$
{}^{\nu}[(\mathbf{E}_1 + \mathbf{E}_2){*}\mathbf{E}_3] = \beta\epsilon^{123k_3\cdots k_\nu}\mathbf{E}_{k_3}\cdots\mathbf{E}_{k_\nu}, \qquad (k_j = 0, 1, \cdots, \nu),
$$

where

$$
\beta^2 = \frac{(\nu-2)!\begin{vmatrix} (\mathbf{E}_1+\mathbf{E}_2):(\mathbf{E}_1+\mathbf{E}_2) & (\mathbf{E}_1+\mathbf{E}_2):\mathbf{E}_3 \\ \mathbf{E}_3:(\mathbf{E}_1+\mathbf{E}_2) & \mathbf{E}_3:\mathbf{E}_3 \end{vmatrix}}{\epsilon^{123j_3\cdots j_\nu}\epsilon^{123k_3\cdots k_\nu}\mathbf{E}_{j_3}:\mathbf{E}_{k_3}\cdots\mathbf{E}_{j_\nu}:\mathbf{E}_{k_\nu}} = 2,
$$

from which, by comparison with (4.7), (4.6) follows immediately.

If we now set $\mathbf{A}_i = \alpha_i^j\mathbf{E}_j$, $(j = 1, \cdots, \nu;\ i = 1, 2, 3)$, we have

$$
{}^{\nu}(\mathbf{A}_1 + \mathbf{A}_2){*}\mathbf{A}_3 = {}^{\nu}\{[(\alpha_1^j + \alpha_2^j)\mathbf{E}_j]{*}(\alpha_3^k\mathbf{E}_k)\}.
$$

By extension of (4.6) and the aid of (ii) the right member of this equation can be written

$$
(\alpha_1^1 + \alpha_2^1)\alpha_3^k\,{}^{\nu}(\mathbf{E}_1{*}\mathbf{E}_k) + \cdots + (\alpha_1^\nu + \alpha_2^\nu)\alpha_3^k\,{}^{\nu}(\mathbf{E}_\nu{*}\mathbf{E}_k).
$$

Whence, with the aid of (i) and (ii),

$$'[(\mathbf{A}_1 + \mathbf{A}_2)*\mathbf{A}_3] = '[(\alpha_1^i\mathbf{E}_i)*(\alpha_3^k\mathbf{E}_k)] + '[(\alpha_2^j\mathbf{E}_j)*(\alpha_3^k\mathbf{E}_k)] = '(\mathbf{A}_1*\mathbf{A}_3) + '(\mathbf{A}_2*\mathbf{A}_3).$$

(iv) and (v) are direct consequences of the definition of a star product.

5. **Mixed products.** The multiple *inner* product of $'(\mathbf{A}_1*\mathbf{A}_2)$, $(2 < \nu \leqq n^p)$, may be taken with any number of polyadics of the same class or of different classes to form *mixed* products. If $\mathbf{A}_3, \cdots, \mathbf{A}_m$, $(m \leqq \nu)$, are polyadics of the same subspace $\mathfrak{P}_\nu^{(p)}$ as $\mathbf{A}_1$ and $\mathbf{A}_2$ we have with the help of (4.1)

$$'(\mathbf{A}_1*\mathbf{A}_2):\mathbf{A}_3 \cdots \mathbf{A}_m = \beta\epsilon^{12i_3\cdots i_\nu}\mathbf{A}_2:\mathbf{B}_{i_3} \cdots \mathbf{A}_m:\mathbf{B}_{i_m}\mathbf{B}_{i_{m+1}} \cdots \mathbf{B}_{i_\nu}, \tag{5.1}$$

$$(i_k = 1, \cdots, \nu),$$

which may be expressed in the determinant form

$$'(\mathbf{A}_1*\mathbf{A}_2):\mathbf{A}_3 \cdots \mathbf{A}_m = \beta \begin{vmatrix} \mathbf{A}_3:\mathbf{B}_3 \cdots \mathbf{A}_3:\mathbf{B}_\nu \\ \mathbf{A}_m:\mathbf{B}_3 \cdots \mathbf{A}_m:\mathbf{B}_\nu \\ \mathbf{B}_2 \quad \cdots \quad \mathbf{B}_\nu \\ \cdots\cdots\cdots\cdots \\ \mathbf{B}_3 \quad \cdots \quad \mathbf{B}_\nu \end{vmatrix} \tag{5.2}$$

It follows from (4.1) and consideration of the determinant of (5.2) that for $m < \nu$

$$\{'(\mathbf{A}_1*\mathbf{A}_2):\mathbf{A}_3 \cdots \mathbf{A}_m\}:\mathbf{A}_i = '(\mathbf{A}_1*\mathbf{A}_2):\mathbf{A}_3 \cdots \mathbf{A}_m\mathbf{A}_i = 0, \qquad (i = 1, \cdots, m),$$

where the $\mathbf{A}_i$ may obviously be applied equally well to the mixed product on any of the free polyadic files indicated by the $\mathbf{B}$'s on the right of (5.1) with the same result. It follows that

THEOREM 5. *If any element $\mathbf{A}_i$ of the product $'(\mathbf{A}_1*\mathbf{A}_2):\mathbf{A}_3 \cdots \mathbf{A}_m$ is repeated or is a linear combination of any of the others, the product will vanish.*

Or, stated geometrically

COROLLARY. *The mixed product $'(\mathbf{A}_1*\mathbf{A}_2):\mathbf{A}_3 \cdots \mathbf{A}_m$ is orthogonal to any of its polyadic factors on each of its $\nu - m$ ordered polyadic files.*

It will be shown later that, if all the polyadics $\mathbf{A}_i$ lie in the subspace over which the product $'(\mathbf{A}_1*\mathbf{A}_2)$ is taken, the mixed product is unaffected by changing the position of the star. Hence in this case there will result no ambiguity in using the notation

$$'(\mathbf{A}_1*\mathbf{A}_2):\mathbf{A}_3 \cdots \mathbf{A}_m = '((\mathbf{A}_1 \cdots \mathbf{A}_m)),$$

where again the ν may be omitted if it is equal to n^p.

Two of these mixed products are of particular interest. If $m = \nu$ we will have a scalar which in the notation of (5.2) becomes

$$'((\mathbf{A}_1 \cdots \mathbf{A}_\nu)) = \beta\,|\,\mathbf{A}_i:\mathbf{B}_j\,|, \qquad (i, j = 3, \cdots, \nu). \tag{5.3}$$

This product will be called the *scalar mixed product* and will often be indicated by enclosing the polyadics $\mathbf{A}_1, \cdots, \mathbf{A}_\nu$ in brackets.

$$(5.4) \qquad {}^{p}((\mathbf{A}_1 \cdots \mathbf{A}_\nu)) = [\mathbf{A}_1 \cdots \mathbf{A}].$$

If $m = \nu - 1$, the resulting mixed product is a polyadic $\mathbf{N}$ which is orthogonal to, and of the same class as the polyadics $\mathbf{A}_i$.

$$(5.5) \qquad {}^{p}((\mathbf{A}_1 \cdots \mathbf{A}_{\nu-1})) = \beta\lambda^i\mathbf{B}_i = \mathbf{N},$$

where λ^i is the cofactor of $\mathbf{A}_\nu : \mathbf{B}_i$ in the determinant $|\mathbf{A}_j : \mathbf{B}_i|$, $(i, j = 3, \cdots, \nu)$. For $p = 1$, $\nu = n = 3$, the latter product becomes the vector product of Gibbs, and the mixed scalar product is his triple scalar product.

To avoid ambiguity the operations star multiplication, indeterminate multiplication, and all types of inner multiplication, shall be performed in the order named. Thus

$$\mathbf{AB}*\mathbf{C} : \mathbf{DF} = [\mathbf{A}(\mathbf{B}*\mathbf{C})] : (\mathbf{DF}).$$

If $\mathbf{E}_1, \cdots, \mathbf{E}_\nu$ is an orthogonal unit basis for a $\mathfrak{P}_\nu^{(p)}$, of which $\mathbf{A}_i$ are a set of elements, we can write $\mathbf{A}_i = \alpha_i^j\mathbf{E}_j$, $(i = 1, \cdots, m \leqq \nu;\ j = 1, \cdots, \nu)$. And with the aid of (4.3)

$$(5.6) \qquad {}^{p}((\mathbf{A}_1 \cdots \mathbf{A}_m)) = \alpha_1^{j_1}\alpha_2^{j_2}\alpha_3^{j_3} \cdots \alpha_m^{j_m}\, {}^{p}(\mathbf{E}_{j_1}*\mathbf{E}_{j_2}) : \mathbf{E}_{j_3} \cdots \mathbf{E}_{j_m},$$

$$(5.7) \qquad {}^{p}((\mathbf{A}_1 \cdots \mathbf{A}_m)) = \epsilon_{j_1 \cdots j_\nu}\alpha_1^{j_1} \cdots \alpha_m^{j_m}\mathbf{E}^{j_{m+1}} \cdots \mathbf{E}^{j_\nu}.$$

This may be expressed in the determinant form

$$(5.8) \qquad {}^{p}((\mathbf{A}_1 \cdots \mathbf{A}_m)) = \begin{vmatrix} \alpha_1^1 & \cdots & \alpha_1^\nu \\ \cdots & \cdots & \cdots \\ \alpha_m^1 & \cdots & \alpha_m^\nu \\ \mathbf{E}^1 & \cdots & \mathbf{E}^\nu \\ \cdots & \cdots & \cdots \\ \mathbf{E}^1 & \cdots & \mathbf{E}^\nu \end{vmatrix}, \qquad (2 \leqq \nu \leqq n^p).$$

THEOREM 6. *Interchanging the elements of a mixed product changes only the sign in accordance with the number of permutations required to make the change. The star may be moved to any position without changing the value of the product.*

The first part of this theorem is obvious due to the properties of the quantities $\epsilon_{j_1 \cdots j_\nu}$. The second part is at once obvious upon realizing that

$$(5.9) \qquad \mathbf{A}_1 \cdots \mathbf{A}_{q-1} : {}^{p}(\mathbf{A}_q*\mathbf{A}_{q+1}) : \mathbf{A}_{q+2} \cdots \mathbf{A}_m = {}^{p}(\mathbf{A}_q*\mathbf{A}_{q+1}) : \mathbf{A}_1 \cdots \mathbf{A}_{q-1}\mathbf{A}_{q+2} \cdots \mathbf{A}_m.$$

Changing the star from between $\mathbf{A}_1$ and $\mathbf{A}_2$ to between $\mathbf{A}_q$ and $\mathbf{A}_{q+1}$ thus serves only to change the rows q and $q + 1$ to rows 1 and 2 in the determinant of (5.8).

If $p = 1$ the polyadics $\mathbf{E}_i$ of (5.8) are replaced by the vectors $\mathbf{e}_i$ and the de-

terminant of (5.8) is identical with the space complement[6] of Naess. As a particular case of (5.8) we note that

$$[\mathbf{E}_1 \cdots \mathbf{E}_\nu] = \begin{vmatrix} 1 & 0 \cdots 0 \\ 0 & 1 \cdots 0 \\ \cdots & \cdots \\ 0 & 0 \cdots 1 \end{vmatrix} = 1. \tag{5.10}$$

THEOREM 7. *The necessary and sufficient condition that m polyadics $\mathbf{A}_i$ be linearly dependent is that*

$$'((\mathbf{A}_1 \cdots \mathbf{A}_m)) = 0. \tag{5.11}$$

The necessary part of this theorem follows directly from Theorem 5. If the multiple inner product is taken of both members of (5.11) with $\mathbf{E}_{k_{m+1}}, \cdots, \mathbf{E}_{k_\nu}$ it follows from (5.7) and (3.8) that

$$\epsilon_{j_1 \cdots j_m k_{m+1} \cdots k_\nu} \alpha_1^{j_1} \cdots \alpha_m^{j_m} = 0$$

That is, all the m-rowed minors of the matrix $\| \alpha_i^j \|$, $(i = 1, \cdots, m;$ $j = 1, \cdots, \nu)$, vanish, which is the condition for the dependence of the $\mathbf{A}_1, \cdots, \mathbf{A}_m$, as shown in Theorem 1.

For $m = \nu$ we see that

$$[\mathbf{A}_1 \cdots \mathbf{A}_\nu] = | \alpha_i^j |, \qquad (i, j = 1, \cdots, \nu). \tag{5.12}$$

6. **Reciprocal polyadic bases, polyadic idemfactors.** If $\mathbf{A}_i$, $(i = 1, \cdots, \nu \leqq n^p)$, are a basis for a $\mathfrak{P}_\nu^{(p)}$ we define the set $\check{\mathbf{A}}^i$ determined by

$$\check{\mathbf{A}}^i = \frac{'((\mathbf{A}_{i+1} \mathbf{A}_{i+2} \cdots \mathbf{A}_\nu \mathbf{A}_1 \cdots \mathbf{A}_{i-1}))}{[\mathbf{A}_i \cdots \mathbf{A}_\nu \mathbf{A}_1 \cdots \mathbf{A}_{i-1}]}, \qquad (i = 1, \cdots, \nu, \text{ not summed}) \tag{6.1}$$

as the *reciprocals* to the set $\mathbf{A}_i$ in $\mathfrak{P}_\nu^{(p)}$. From Theorem 5 it follows that

$$\check{\mathbf{A}}^i : \mathbf{A}_j = \delta_j^i. \tag{6.2}$$

It is evident that the set $\mathbf{E}_i$ is reciprocal to itself. Its elements may be written with either upper or lower indices.

If the values of $\check{\mathbf{A}}^i$ be obtained by means of (6.1) and (5.7) and the results substituted in $[\check{\mathbf{A}}^1 \cdots \check{\mathbf{A}}^\nu]$ we find after simplification that

$$[\check{\mathbf{A}}^1 \cdots \check{\mathbf{A}}^\nu][\mathbf{A}_1 \cdots \mathbf{A}_\nu] = 1. \tag{6.3}$$

It follows from Theorem 7 that the elements of the set $\check{\mathbf{A}}^i$ are independent if the $\mathbf{A}_i$ are independent, and hence form a basis for $\mathfrak{P}_\nu^{(p)}$. It is obvious that the set $\mathbf{A}_i$ is also reciprocal to the set $\check{\mathbf{A}}^i$.

Let $\mathbf{A}_1, \cdots, \mathbf{A}_N$ be a basis for a polyadic space $\mathfrak{P}^{(p)}$, where $N = n^p$, and $\mathbf{M}$ be any polyadic in $\mathfrak{P}^{(p)}$. Then we can write

$$\mathbf{M} = m^i \mathbf{A}_i, \qquad (i = 1, \cdots, N).$$

[6] Loc. cit. Note 2.

If the multiple inner product of both members of this equation be taken with $\bar{\mathbf{A}}^j$ we have the N equations

$$m^j = \mathbf{M}:\bar{\mathbf{A}}^j$$

which, if substituted back in the equation above, gives

$$\mathbf{M} = \mathbf{M}:\bar{\mathbf{A}}^i\mathbf{A}_i .$$

Similarly, if $\mathbf{M}$ be expressed in terms of $\bar{\mathbf{A}}^i$ and the resulting equation be multiplied by $\mathbf{A}_j$:, we would arrive at the relation

$$\mathbf{M} = \mathbf{M}:\mathbf{A}_i\bar{\mathbf{A}}^i.$$

The polyadic

$$\mathbf{I}^{(2p)} = \bar{\mathbf{A}}^i\mathbf{A}_i = \mathbf{A}_i\bar{\mathbf{A}}^i, \qquad (i = 1, \cdots, N), \tag{6.4}$$

evidently constitutes an *idemfactor* with respect to multiple inner multiplication for all polyadics of class p. If $\mathbf{M}$ be of class greater than p, we can use (2.4) to write $\mathbf{M} = \mathbf{B}_i^{(p)}\mathbf{C}^i$.
Whence

$$\mathbf{I}^{(2p)}:{}_p\mathbf{M} = (\mathbf{I}^{(2p)}:\mathbf{B}_i^{(p)})\mathbf{C}^i = \mathbf{B}_i^{(p)}\mathbf{C}^i = \mathbf{M}, \tag{6.5}$$

$$\mathbf{I}^{(2p)}:\mathbf{M} = (\mathbf{I}^{(2p)}:\mathbf{B}_i^{(p)}):\mathbf{C}^i = \mathbf{B}_i^{(p)}:\mathbf{C}^i. \tag{6.6}$$

THEOREM 8. *The p-tuple inner product of $\mathbf{I}^{(2p)}$ with any polyadic $\mathbf{M}$ of class p or greater yields the polyadic $\mathbf{M}$ itself. If $\mathbf{M}$ is of class $p + q$ its $2p$-tuple inner product with $\mathbf{I}^{(2p)}$ yields a polyadic of class $| p - q |$ formed by taking the multiple inner product of the first p factors with the remaining factors in each polyad of $\mathbf{M}$.*

To investigate idemfactors for subspaces of $\mathfrak{P}^{(p)}$ suppose that $\mathbf{A}_1, \cdots, \mathbf{A}_\nu$ constitute a basis for a $\mathfrak{P}_\nu^{(p)}$ and $\mathbf{B}_1, \cdots, \mathbf{B}_\mu$, $(\mu = N - \nu)$, are a basis for a subspace $\mathfrak{P}_\mu^{(p)}$; and assume further that

$$\mathbf{A}_i:\mathbf{B}_\alpha = 0, \qquad (i = 1, \cdots, \nu; \alpha = 1, \cdots, \mu).$$

Then every element $\mathbf{K}$ of $\mathfrak{P}_\nu^{(p)}$ is orthogonal to every element $\mathbf{L}$ of $\mathfrak{P}_\mu^{(p)}$ and $\mathfrak{P}_\mu^{(p)}$ constitutes the orthogonal complement of $\mathfrak{P}_\nu^{(p)}$. The polyadics $\bar{\mathbf{A}}^i$ reciprocal to the set $\mathbf{A}_i$ in $\mathfrak{P}_\nu^{(p)}$ and the $\bar{\mathbf{B}}^\alpha$ reciprocal to the set $\mathbf{B}_\alpha$ in $\mathfrak{P}_\mu^{(p)}$ satisfy the relations

$$\mathbf{A}_i:\bar{\mathbf{A}}^j = \delta_i^j, \qquad \bar{\mathbf{A}}^i:\mathbf{B}_\alpha = \mathbf{A}_i:\bar{\mathbf{B}}^\alpha = 0, \qquad \mathbf{B}_\alpha:\bar{\mathbf{B}}^\beta = \delta_\alpha^\beta .$$

It follows that the set $\bar{\mathbf{A}}^1, \cdots, \bar{\mathbf{A}}^\nu, \bar{\mathbf{B}}^1, \cdots, \bar{\mathbf{B}}^\mu$ is reciprocal to the set $\mathbf{A}_1, \cdots, \mathbf{A}_\nu, \mathbf{B}_1, \cdots, \mathbf{B}_\mu$ in $\mathfrak{P}^{(p)}$.

The idemfactors for $\mathfrak{P}^{(p)}$, $\mathfrak{P}_\nu^{(p)}$, and $\mathfrak{P}_\mu^{(p)}$ are, respectively,

$$\mathbf{I}^{(2p)} = \mathbf{A}_i\bar{\mathbf{A}}^i + \mathbf{B}_\alpha\bar{\mathbf{B}}^\alpha, \qquad {}^\nu\mathbf{I}^{(2p)} = \mathbf{A}_i\bar{\mathbf{A}}^i, \qquad {}^\mu\mathbf{I}^{(2p)} = \mathbf{B}_\alpha\bar{\mathbf{B}}^\alpha,$$

$$(i = 1, \cdots, \nu; \alpha = 1, \cdots, \mu),$$

from which it follows that

$${}^\nu\mathbf{I} = \mathbf{I} - \mathbf{B}_\alpha\bar{\mathbf{B}}^\alpha, \qquad (\alpha = 1, \cdots, N - \nu; N = n^p). \tag{6.7}$$

For any $\mathbf{K}$ in $\mathfrak{P}_\nu^{(p)}$ and $\mathbf{L}$ in $\mathfrak{P}_\mu^{(p)}$,

$$\mathbf{K} = \kappa^i \mathbf{A}_i , \qquad \mathbf{L} = \lambda^\alpha \mathbf{B}_\alpha ,$$

we see that

$$'\mathbf{I}:\mathbf{K} = (\mathbf{I} - \mathbf{B}_\alpha \bar{\mathbf{B}}^\alpha):(\kappa^i \mathbf{A}_i) = \mathbf{K}, \qquad '\mathbf{I}:\mathbf{L} = (\mathbf{I} - \mathbf{B}_\alpha \bar{\mathbf{B}}^\alpha):(\lambda^\beta \mathbf{B}_\beta) = 0;$$

and for any $\mathbf{Q}$ in $\mathfrak{P}^{(p)}$, i.e., $\mathbf{Q} = \kappa^i \mathbf{A}_i + \lambda^\beta \mathbf{B}_\beta$,

$$'\mathbf{I}:\mathbf{Q} = (\mathbf{I} - \mathbf{B}_\alpha \bar{\mathbf{B}}^\alpha):(\kappa^i \mathbf{A}_i + \lambda^\beta \mathbf{B}_\beta) = \kappa^i \mathbf{A}_i .$$

From which we conclude

THEOREM 9. *The multiple inner product of the idemfactor* $'\mathbf{I}^{(2p)}$ *of any subspace* $\mathfrak{P}_\nu^{(p)}$ *with any polyadic* $\mathbf{Q}$ *of order* p *yields the projection of* $\mathbf{Q}$ *upon* $\mathfrak{P}_\nu^{(p)}$.

To obtain the form of the idemfactor $\mathbf{I}^{(2p)}$ in terms of basis vectors, we let $\mathbf{a}_{1,\alpha_1}, \cdots, \mathbf{a}_{p,\alpha_p}$ be any p sets of such basis vectors and select for the polyadics $\mathbf{A}_i$ of (6.4) the p-ads $\mathbf{a}_{1,\alpha_1} \cdots \mathbf{a}_{p,\alpha_p}$. Then the $\bar{\mathbf{A}}^i$ will be the p-ads $\mathbf{a}_1^{\alpha_1} \cdots \mathbf{a}_p^{\alpha_p}$, and

$$\mathbf{I}^{(2p)} = \mathbf{a}_{1,\alpha_1} \cdots \mathbf{a}_{p,\alpha_p} \mathbf{a}_1^{\alpha_1} \cdots \mathbf{a}_p^{\alpha_p} .$$

Evidently a mixed set of the p-ads containing both upper and lower indices could have been selected for $\mathbf{A}_i$, in which case the $\bar{\mathbf{A}}^i$ would be the corresponding oppositely indexed p-ads. Thus for $\mathbf{I}^{(4)}$ we have

$$\mathbf{I}^{(4)} = \mathbf{a}_\alpha \mathbf{b}_\beta \mathbf{a}^\alpha \mathbf{b}^\beta = \mathbf{a}^\alpha \mathbf{b}^\beta \mathbf{a}_\alpha \mathbf{b}_\beta = \mathbf{a}_\alpha \mathbf{b}^\beta \mathbf{a}^\alpha \mathbf{b}_\beta = \mathbf{a}^\alpha \mathbf{b}_\beta \mathbf{a}_\alpha \mathbf{b}^\beta . \tag{6.8}$$

7. **Multiple multiplex products.** We have used the double dot notation to indicate multiple inner products on the simple indices, i.e., on the vector factors of the polyads determining two polyadics. The idea will now be generalized to indicate multiple inner products on the multiplex indices, i.e., on the polyadic factors which make up the terms of two polyadics under decompositions such as (2.3) and (2.4). Such a product will be indicated by $\circ$ interposed between the two polyadics, with an appropriate subscript $\circ_t$ if the number of polyadic factors for which the multiple inner product is taken is limited or is to be emphasized. For convenience, we will call this multiple multiplex inner product a *double circle* product.

Thus if the three polyadics $\mathbf{A}^{(9)}$, $\mathbf{B}^{(5)}$, $\mathbf{C}^{(7)}$, are expressed with the aid of (2.4) in the forms

$$\mathbf{A}^{(9)} = \mathbf{P}_{(3)}^{ij} \mathbf{P}_j^{(4)} \mathbf{P}_j^{(2)}, \qquad \mathbf{B}^{(5)} = \mathbf{Q}_k^{(2)} \mathbf{Q}_{(3)}^k , \qquad \mathbf{C}^{(7)} = \mathbf{R}_{(3)}^s \mathbf{R}_s^{(4)},$$

we write

$$\begin{aligned} \mathbf{A} \circ \mathbf{B} &= (\mathbf{P}_{(3)}^{ij} \mathbf{P}_i^{(4)} \mathbf{P}_j^{(2)}) \circ (\mathbf{Q}_k^{(2)} \mathbf{Q}_{(3)}^k) = \mathbf{P}_{(3)}^{ij} : \mathbf{Q}_k^{(2)} \mathbf{P}_i^{(4)} : \mathbf{Q}_{(3)}^k \mathbf{P}_j^{(2)}; \\ \mathbf{A} \circ \mathbf{B} \mathbf{C} &= \mathbf{P}_{(3)}^{ij} : \mathbf{Q}_k^{(2)} \mathbf{P}_i^{(4)} : \mathbf{Q}_{(3)}^k \mathbf{P}_j^{(2)} : \mathbf{R}_{(3)}^s \mathbf{R}_s^{(4)}. \end{aligned} \tag{7.1}$$

This product bears the same relation to the double dot product as the double dot product does to the single dot product. If the classes of the polyadic factors of $\mathbf{A}$ are, respectively, equal to the classes of the polyadic factors of $\mathbf{B}$, then the product $\mathbf{A} \circ \mathbf{B}$ is equivalent to $\mathbf{A}:\mathbf{B}$. Like double dot products, double circle

products are commutative except in the case where the number of polyadic factors to which the double circle product is applied is limited. (Compare with Section 3).

In a like manner we indicate by a double star between two polyadics the product formed by taking the multiplex outer products of the polyadic factors of the two polyadics in the order in which they occur, always starting from the left. For example, if **A** and **B** are two polyadics of the same order with the decompositions

$$\mathbf{A} = \mathbf{P}_{(c_1)}^{i_2\cdots i_t}\mathbf{P}_{i_2}^{(c_2)}\cdots\mathbf{P}_{i_t}^{(c_t)}, \qquad \mathbf{B} = \mathbf{Q}_{(c_1)}^{j_2\cdots j_t}\mathbf{Q}_{j_2}^{(c_2)}\cdots\mathbf{Q}_{j_t}^{(c_t)},$$

then the double star product of **A** and **B** on the multipartite indices $u_1, \cdots, u_t$ consisting of $c_1, \cdots, c_t$ indices, respectively, is defined by

$$(7.2) \qquad {}^{c_1\cdots c_t}(\mathbf{A}\overset{*}{*}\mathbf{B}) = (\mathbf{P}_{(c_1)}^{i_2\cdots i_t}{*}\mathbf{Q}_{(c_1)}^{j_2\cdots j_t})(\mathbf{P}_{i_2}^{(c_2)}{*}\mathbf{Q}_{j_2}^{(c_2)})\cdots(\mathbf{P}_{i_t}^{(c_t)}{*}\mathbf{Q}_{j_t}^{(c_t)}),$$

where each star product on the right is to be interpreted as in (4.1), the k-th, $(k = 1, \cdots, t)$, star product being taken over the polyadic space $\mathfrak{P}^{(c_k)}$ of dimensionality n^{c_k}. If the classes c_i are all equal the pre-superscript $c_1\cdots c_t$ will be omitted or replaced by the single letter t.

It is obvious from the properties (4.5) of the single star product that, for any decomposition on any plyadics **A**, **B** and **C** of a polyadic space $\mathfrak{P}^{(p)}$, $\left(p = \sum_{i=1}^{t} c_i\right)$, made in any way on t partitions consisting of $c_1, \cdots, c_t$ indices, respectively, and for all real scalars ξ and η, the following laws are valid:

$$(7.3) \qquad \begin{aligned} (\xi+\eta)\,{}^{c_1\cdots c_t}(\mathbf{A}\overset{*}{*}\mathbf{B}) &= \xi\,{}^{c_1\cdots c_t}(\mathbf{A}\overset{*}{*}\mathbf{B}) + \eta\,{}^{c_1\cdots c_t}(\mathbf{A}\overset{*}{*}\mathbf{B}), && \text{(i)}\\ {}^{c_1\cdots c_t}[(\xi\mathbf{A})\overset{*}{*}(\eta\mathbf{B})] &= \xi\eta\,{}^{c_1\cdots c_t}(\mathbf{A}\overset{*}{*}\mathbf{B}), && \text{(ii)}\\ {}^{c_1\cdots c_t}[(\mathbf{A}+\mathbf{B})\overset{*}{*}\mathbf{C}] &= {}^{c_1\cdots c_t}(\mathbf{A}\overset{*}{*}\mathbf{C}) + {}^{c_1\cdots c_t}(\mathbf{B}\overset{*}{*}\mathbf{C}), && \text{(iii)}\\ {}^{c_1\cdots c_t}(\mathbf{A}\overset{*}{*}\mathbf{B}) &= (-1)^t\,{}^{c_1\cdots c_t}(\mathbf{B}\overset{*}{*}\mathbf{A}). && \text{(iv)} \end{aligned}$$

It is obvious from the uniqueness of the single star product and its properties (4.5) that not the particular decompositions chosen, but the vector files selected to constitute each partition determine the double star product; that is,

THEOREM 10. *The double star product of two polyadics formed on a given set of multipartite indices is unique.*

The uniqueness of the double circle product follows from the well established uniqueness of the inner products.

8. **Mixed multiple multiplex products.** Combinations of double star with double dot or double circle products may be built up in numerous ways to form polyadic products of various classes. Of particular interest are the mixed double star, double circle products formed from a single polyadic.[7] We will

[7] These functions are a generalization of Gibbs' "second" and "third" of a dyadic, Gibbs-Wilson, *Vector Analysis*, p. 310.

have frequent use of these in the sequel and will indicate them, after removing a factor $m!$, by the abbreviated notation $(\mathbf{A})_m^{c_1 \cdots c_t}$

$$(8.1) \qquad (\mathbf{A})_m^{c_1 \cdots c_t} = \frac{1}{m!}\, {}^{c_1 \cdots c_t}(\mathbf{A}^{*}_{*_t} \mathbf{A}) \circ \mathbf{A}^{m-2}, \qquad (2 \leqq m \leqq k),$$

where the exponent $m - 2$ indicates that $\mathbf{A}$ occurs $m - 2$ times, k is the least of the numbers n^{c_i}, and $c_1 \cdots c_t$ indicates the class of the decomposition polyadics of $\mathbf{A}$. That is, $\mathbf{A}$ is to be taken as the sum of products of polyadics of classes $c_1, \cdots, c_t$. The order of operations in determining these products is the same as that for simple mixed products stated in Section 5.

If t is odd these functions of the polyadic $\mathbf{A}$ will be found to vanish identically. This can be prevented by making an odd number of the mixed simple products occurring in the expansion non-signant. To simplify the discussion we will suppose that the first such product alone is non-signant. This will be indicated by taking the $(t - 1)$-th power of the sign of the first such mixed product in each term of the expansion. The double star with the subscripted t is always to have this significance if t is odd.

Thus if we use the form (2.4) for $\mathbf{A}$,

$$\mathbf{A} = \mathbf{D}_{(c_1)}^{u_1 \cdots u_t} \mathbf{D}_{u_1}^{(c_1)} \cdots \mathbf{D}_{u_t}^{(c_t)}, \qquad (u_k = 1, \cdots, n^{c_k}),$$

$$(8.2) \quad (\mathbf{A})_m^{c_1 \cdots c_t} = \frac{1}{m!} ((\pm 1)^{t-1} (\mathbf{D}_{(c_1)}^{u_{11} \cdots u_{1t}} \cdots \mathbf{D}_{(c_1)}^{u_{m1} \cdots u_{mt}})) ((\mathbf{D}_{u_{11}}^{(c_2)} \cdots \mathbf{D}_{u_{m1}}^{(c_2)})) \cdots ((\mathbf{D}_{u_{1t}}^{(c_t)} \cdots \mathbf{D}_{u_{mt}}^{(c_t)})), \qquad (m = 2, \cdots, k).$$

It is obvious, since the number of signed factors in each term of (8.2) is even, that the order of the polyadics in the simple mixed products in the right number may simultaneously be changed without altering the value of the expression. If the classes c_i of the decomposition polyadics are all the same, say c, then the class of $\mathbf{A}$ is $p = ct$, and we have the special cases of the functions $(\mathbf{A})^{c_1 \cdots c_t}$, which we designate by $(\mathbf{A})_m^t$,

$$(8.3) \qquad (\mathbf{A})_m^t = (\mathbf{A})_m^{c \cdots c} = \frac{1}{m!}\, \mathbf{A}^{*}_{*} \mathbf{A} \circ \mathbf{A}^{m-2}.$$

From the uniqueness of the double star and the double circle products under all decompositions on given sets of indices, it follows that the functions $(\mathbf{A})_m^{c_1 \cdots c_t}$ are multiple polyadic invariants[8] of $\mathbf{A}$ on the multipartite indices $u_1, \cdots, u_t$ consisting of the $c_1, \cdots, c_t$ indices, respectively.

It is to be recalled that in considering the various multiplex products discussed in Sections 7 and 8 we are confining ourselves, as pointed out in the beginning of Section 2, to the consideration of that isomer of $\mathbf{A}$ which permits of representation in forms such as (2.3) and (2.4), where the partitions $u_1, \cdots, u_t$ are made on consecutive indices. Actually there are as many functions of the form (8.3),

[8] The terminology is that of Hitchcock. Cf. *Multiple invariants of a* p*-way matrix or tensor*, Jour. Math. and Physics, Vol. 7 (1927), p. 40.

for instance, for a given t and m, as there are different ways in which the vector files of $\mathbf{A}$ may be grouped into t sets each containing c files. To attempt to indicate these would simply add to the difficulties of notation and contribute nothing to the generality of the theory. The same remark is to be made regarding all functions of polyadics in the pages which follow which depend upon decompositions on multipartite indices.

9. **Multiple invariants of polyadics.** The functions $(\mathbf{A})_m^t$ defined in (8.3) are polyadics of class $(n^c - m)c$, where $c = p/t$. Suppose t to be *even*, and let $\mathbf{I}^{(p)}$ be the idemfactor for the polyadic space $\mathfrak{P}^{(\frac{p}{2})}$. $\mathbf{I}^{(p)}$ is then of the same class as $\mathbf{A}$, namely ct. Let the same partitions be selected for the decomposition to be made on $\mathbf{I}$ as on $\mathbf{A}$. By taking the double circle product of $n^c - m$ such idemfactors with the functions $(\mathbf{A})_m^t$ we obtain several sets of multiple scalar invariants of $\mathbf{A}$.

$$
\begin{aligned}
[\mathbf{A}]_m^t &= \frac{1}{(k-m)!}\,(\mathbf{A})_m^t \circ \mathbf{I}_{(p)}^{k-m} \\
&= \frac{1}{m!(k-m)!}\,\mathbf{A}_{*t}^{*}\,\mathbf{A}\circ\mathbf{A}^{m-2}\mathbf{I}_{(p)}^{k-m}, \qquad (m = 2, \cdots, k = n^c).
\end{aligned} \tag{9.1}
$$

In addition to the $n^c - 1$ invariants thus defined for each possible value of t, i.e., the even factors of p, there exists another invariant, which is the same for all values of t. It is given by

$$
[\mathbf{A}]_1^t = \mathbf{A}{:}\mathbf{I}^{(p)} = A^{\alpha_1\cdots\alpha_q\alpha_1\cdots\alpha_q}, \qquad (q = p/2), \tag{9.2}
$$

where $A^{\alpha_1\cdots\alpha_p}$ are the components of $\mathbf{A}$ with respect to the basis vectors $\mathbf{e}_1\cdots\mathbf{e}_p$, the right member being obtained with the aid of Theorem 8.

Of special interest are the invariants of the set (9.1) formed when $m = k = n^c$. These exist when t is either *odd* or *even*. To study these invariants more closely we will employ the form (2.3) for $\mathbf{A}$ using the polyadics $\mathbf{E}_u^{(c)}$ as a basis.

$$
\mathbf{A} = A^{u_1\cdots u_t}\mathbf{E}_{u_1}\cdots\mathbf{E}_{u_t}, \qquad (u_i = 1, \cdots, k = n^c).
$$

Then

$$
\begin{aligned}
[\mathbf{A}]_k^t = (\mathbf{A})_k^t &= \frac{1}{k!}\,\mathbf{A}_{*t}^{*}\,\mathbf{A}\circ\mathbf{A}^{k-2}, \\
&= \frac{1}{k!}\,A^{u_{11}\cdots u_{1t}}\cdots A^{u_{k1}\cdots u_{kt}}[\mathbf{E}_{u_{11}}\cdots\mathbf{E}_{u_{k1}}]^{t-1} \\
&\qquad [\mathbf{E}_{u_{12}}\cdots\mathbf{E}_{u_{k2}}]\cdots[\mathbf{E}_{1t}\cdots\mathbf{E}_{kt}],
\end{aligned} \tag{9.3}
$$

which with the aid of (5.10) can be written

$$
[\mathbf{A}]_k^t = \frac{1}{k!}\,(\epsilon_{u_{11}\cdots u_{k1}})^{t-1}\epsilon_{u_{12}\cdots u_{k2}}\cdots\epsilon_{u_{1t}\cdots u_{kt}}A^{u_{11}\cdots u_{1t}}A^{u_{21}\cdots u_{2t}}\cdots A^{u_{k1}\cdots u_{kt}},
$$

or

$$[\mathbf{A}]_k^t = |A^{u_1 \cdots u_t}|, \quad (k = n^c; c = pt), \tag{9.4}$$

where the right member is a t-way determinant[9] of order n^c which is full signed if t is even and non-signant on the first index if t is odd.

As an example of multiple invariants consider those of a polyadic $\mathbf{A}$ of class 12. There exist four sets of such invariants, namely

$$[\mathbf{A}]_{m_1}^2, \quad [\mathbf{A}]_{m_2}^4, \quad [\mathbf{A}]_{m_3}^6, \quad [\mathbf{A}]_{m_4}^{12}$$

where $m_1 = 1, \cdots, n^6$, $m_2 = 1, \cdots, n^3$, $m_3 = 1, \cdots, n^2$, $m^4 = 1, \cdots, n$, the invariant for $m_i = 1$, $(i = 1, \cdots, 4)$, being common to all four sets.

10. **Characteristic Determinants.** We will now show that

THEOREM 11. *If $t = p/c$ is even the invariants* (9.1), (9.2) *are the coefficients of the powers of* λ, *except for numerical factors, in the expansion of the t-way full signed determinant*

$$|A^{u_1 \cdots u_t} - \lambda \delta_{u_{q+1}}^{u_1} \cdots \delta_{u_t}^{u_q}|, \quad (q = t/2; u_j = 1, \cdots, n^c), \tag{10.1}$$

where each partition u_i consists of c indices.

We will call this the *characteristic* determinant of $\mathbf{A}^{(p)}$ on the equal partitions $u_1, \cdots, u_t$. Every polyadic has as many such characteristic determinants as its class has distinct even factors.[10]

To obtain a decomposition of the idemfactor $\mathbf{I}^{(p)}$ we use the method of Section 3 to write

$$\mathbf{I} = \mathbf{e}_{\alpha_1} \cdots \mathbf{e}_{\alpha_s} \mathbf{e}_{\alpha_1} \cdots \mathbf{e}_{\alpha_s} = \mathbf{E}_{u_1} \cdots \mathbf{E}_{u_q} \mathbf{E}_{u_1} \cdots \mathbf{E}_{u_q}, \quad (2s = ct = p; 2q = t),$$

where

$$\mathbf{E}_1 = \mathbf{e}_1 \cdots \mathbf{e}_c, \quad \mathbf{E}_2 = \mathbf{e}_1 \cdots \mathbf{e}_{c-2} \mathbf{e}_c \mathbf{e}_{c-1}, \quad \text{etc.}$$

Hence

$$\mathbf{I} = \delta_{u_{q+1}}^{u_1} \cdots \delta_{u_t}^{u_q} \mathbf{E}_{u_1} \cdots \mathbf{E}_{u_q} \mathbf{E}_{u_{q+1}} \cdots \mathbf{E}_{u_t}. \tag{10.2}$$

It follows from (9.3) that

$$|A^{u_1 \cdots u_t} - \lambda \delta_{u_{q+1}}^{u_1} \cdots \delta_{u_t}^{u_q}| = [\mathbf{A} - \lambda \mathbf{I}]_k^t, \quad (k = n^c). \tag{10.3}$$

From (9.3) we have

$$[\mathbf{A} - \lambda \mathbf{I}]_k^t = \frac{1}{k!} (\mathbf{A} - \lambda \mathbf{I})_{*_1}^{*} (\mathbf{A} - \lambda \mathbf{I})_{*_2}^{*} \cdots (\mathbf{A} - \lambda \mathbf{I})^{k-2}.$$

[9] Consult Rice, L. H., *Introduction to higher determinants*, Jour. Math. and Physics, Vol. 9 (1929), pp. 47–69.

[10] That is, every polyadic has this number for every insomer of $\mathbf{A}$ wherein at least one of the partitions u_i does not contain the same set of indices. See last paragraph of Sec. 8.

With the aid of Theorem 6 and the properties (7.3) of double star products the right member can be expanded so that

$$k![\mathbf{A} - \lambda\mathbf{I}]_k^t = \mathbf{A}^*_{*_t}\mathbf{A}^\circ_\circ\mathbf{A}^{k-2} + \sum_{r=1}^{k-2}(-1)^r\binom{k}{r}\mathbf{A}^*_{*_t}\mathbf{A}^\circ_\circ\mathbf{A}^{k-r-2}\mathbf{I}^r_{(p)}\lambda^r + (-1)^{k-1}k\mathbf{A}^*_{*_t}\mathbf{I}_\circ\mathbf{I}^{k-2}\lambda^{k-1} + (-1)^k\mathbf{I}^*_{*_t}\mathbf{I}^\circ_\circ\mathbf{I}^{k-2}\lambda^k.$$

And with the aid of (9.1)

$$[\mathbf{A} - \lambda\mathbf{I}]_k^t = [\mathbf{A}]_k^t + \sum_{r=1}^{k-2}(-1)^r[\mathbf{A}]_{k-r}^t\lambda^r + (-1)^{k-1}\mathbf{A}^\circ_\circ(\mathbf{I})_{k-1}^t\lambda^{k-1} + (-1)^k[\mathbf{I}]_k^t\lambda^k. \tag{10.4}$$

To determine $[\mathbf{I}]_k^t$ we have

$$[\mathbf{I}]_k^t = \frac{1}{k!}[\mathbf{E}_{u_{11}}\cdots\mathbf{E}_{u_{1k}}]\cdots[\mathbf{E}_{u_{q1}}\cdots\mathbf{E}_{u_{qk}}][\mathbf{E}_{u_{11}}\cdots\mathbf{E}_{u_{1k}}]\cdots[\mathbf{E}_{u_{q1}}\cdots\mathbf{E}_{u_{qk}}], \qquad (q = t/2);$$

and with the aid of (5.10)

$$[\mathbf{I}]_k^t = \frac{1}{k!}\epsilon_{u_{11}\cdots u_{1k}\cdots\epsilon_{u_{q1}\cdots u_{qk}}}\epsilon_{u_{11}\cdots u_{1k}}\cdots\epsilon_{u_{q1}\cdots u_{qk}},$$

$$= \frac{1}{k!}(k!)^m = (k!)^{m-1}.$$

Also

$$(\mathbf{I})_{k-1}^t = \frac{1}{(k-1)!}((\mathbf{E}_{u_{1,1}}\cdots\mathbf{E}_{u_{1,k-1}}))\cdots((\mathbf{E}_{u_{q,1}}\cdots\mathbf{E}_{u_{q,k-1}}))((\mathbf{E}_{u_{1,1}}\cdots\mathbf{E}_{u_{1,k-1}}))\cdots((\mathbf{E}_{u_{q,1}}\cdots\mathbf{E}_{u_{q,k-1}})),$$

$$= \frac{1}{(k-1)!}\epsilon_{u_{11}\cdots u_{1k}}\mathbf{E}_{u_{1k}}\cdots\epsilon_{u_{q1}\cdots u_{qk}}\mathbf{E}_{u_{qk}}\epsilon_{u_{11}\cdots u_{1k}}\mathbf{E}_{u_{1k}}\cdots\cdots\epsilon_{u_{q1}\cdots u_{qk}}\mathbf{E}_{u_{qk}},$$

$$= [(k-1)!]^{q-1}\mathbf{E}_{u_{1k}}\cdots\mathbf{E}_{u_{qk}}\mathbf{E}_{u_{1k}}\cdots\mathbf{E}_{u_{qk}} = [(k-1)!]^{q-1}\mathbf{I}.$$

Substituting these values in equation (10.4) and applying (9.2)

$$|A^{u_1\cdots u_t} - \lambda\delta^{u_1}_{u_{q+1}}\cdots\delta^{u_q}_{u_t}| = [\mathbf{A}]_k^t + \sum_{r=1}^{k-2}(-1)^r[\mathbf{A}]_{k-r}^t\lambda^r + (-1)^{k-1}[(k-1)!]^{\frac{t-2}{2}}[\mathbf{A}]_1^t\lambda^{k-1} + (-1)^k(k!)^{\frac{t-2}{2}}\lambda^k, \tag{10.5}$$

where in the right member the coefficients of the powers of λ are, within a constant factor, the invariants of (9.1) and (9.2).

11. **Ranks of polyadics and their matrices.** In the discussion of the ranks associated with polyadics we will adopt the following

DEFINITION. A t-way matrix will be said to be of rank r if it contains at

least one r-rowed t-way determinant which is not zero, while all determinants of higher order than r vanish. In evaluating these determinants all indices are to be considered as signed except when t is odd in which case the first index is to be non-signant.

Let the partitions $u_1, \cdots, u_t$ be made on the indices of $A^{\alpha_1 \cdots \alpha_p}$ where the number of indices in each partition may or may not be the same. The $A^{\alpha_1 \cdots \alpha_p}$ are here the components of the polyadic $\mathbf{A}$ with respect to any sets of basis vectors.

DEFINITION. The t-way multipartite rank[11] of $\mathbf{A}$ on the partitions $u_1, \cdots, u_t$ is the rank of the t-way matrix $\| A^{u_1 \cdots u_t} \|$.

We will now show that

THEOREM 12. *The necessary and sufficient condition that the polyadic* $\mathbf{A}$ *be of simple rank* r *on the* m*-th index is that*

$$(\mathbf{A}')_{r+1}^{1,p-1} = 0, \qquad (\mathbf{A}')_{r}^{1,p-1} \neq 0,$$

where $\mathbf{A}'$ *is that isomer given by*

$$\mathbf{A}' = A^{\alpha_m \alpha_1 \cdots \alpha_{m-1} \alpha_{m+1} \cdots \alpha_p} \mathbf{a}_{m,\alpha_m} \mathbf{a}_{1,\alpha_1} \cdots \mathbf{a}_{m-1,\alpha_{m-1}} \mathbf{a}_{m+1,\alpha_{m+1}} \cdots \mathbf{a}_{p,\alpha_p},$$

$$(\alpha_k = 1, \cdots, n).$$

PROOF. Making a partition u of all the indices of $\mathbf{A}'$ save the first $\mathbf{A}'$ may obviously be expressed in the form

$$\mathbf{A}' = A^{\alpha u} \mathbf{a}_\alpha \mathbf{C}_u, \qquad (u = 1, \cdots, k = n^{p-1}),$$

where for convenience of notation we have placed $\mathbf{a}_{m,\alpha_m} = \mathbf{a}_\alpha$. Whence by (8.2)

$$(\mathbf{A}')_m^{1,p-1} = \frac{1}{m!}\, {}^{1,p-1}(\mathbf{A}' *_2 \mathbf{A}') {}^{\circ}_{\circ} \mathbf{A}'^{m-2}$$

$$= \frac{1}{m!} A^{\alpha_1 u_1} \cdots A^{\alpha_m u_m} ((\mathbf{a}_{\alpha_1} \cdots \mathbf{a}_{\alpha_m}))((\mathbf{C}_{u_1} \cdots \mathbf{C}_{u_m})).$$

Now if we let

$$\mathbf{a}_{\alpha_\rho} = a^i_{\alpha_\rho} \mathbf{e}_i, \qquad \mathbf{C}_{u_k} = C^j_{u_k} \mathbf{E}_j, \qquad (i = 1, \cdots, n;\ j = 1, \cdots, k = n^{p-1}),$$

we have with the aid of (5.7)

$$(\mathbf{A}')_m^{1,p-1} = \frac{1}{m!} \epsilon_{i_1 \cdots i_n} \epsilon_{j_1 \cdots j_k} a^{i_1}_{\alpha_1} \cdots a^{i_m}_{\alpha_m} C^{j_1}_{u_1} \cdots C^{j_m}_{u_m} A^{\alpha_1 u_1} \cdots A^{\alpha_m u_m} \mathbf{e}^{i_{m+1}} \cdots \mathbf{e}^{i_n} \mathbf{E}^{j_{m+1}} \cdots \mathbf{E}^{j_k}. \tag{11.1}$$

It follows from the independence of the unitary polyadics $\mathbf{e}^i$ and $\mathbf{E}^j$ that (11.1) vanishes if, and only if, their coefficients

$$\frac{1}{m!} \epsilon_{i_1 \cdots i_n} \epsilon_{j_1 \cdots j_k} a^{i_1}_{\alpha_1} A^{\alpha_1 u_1} C^{j_1}_{u_1} \cdots a^{i_m}_{\alpha_m} A^{\alpha_m u_m} C^{j_m}_{u_m}, \tag{11.2}$$

[11] This includes the case where u_m contains but one index and $t = 2$, in which case we speak of the simple rank of $\mathbf{A}$ on the index α_m.

vanish. But except possibly for sign, these are the m-rowed determinant minors of the matrix $\| a^i_\alpha A^{\alpha u} C^j_u \|$, which may be factored into the three matrices $\| a^i_\alpha \|$, $\| A^{\alpha u} \|$, $\| C^j_u \|$. Since the $\mathbf{a}_\alpha$, $(\alpha = 1 \cdots, n)$, are independent and the $\mathbf{C}_u$, $(u = 1, \cdots, n^{p-1})$, are independent none of the determinant minors of $\| a^i_\alpha \|$ or $\| C^j_u \|$ vanish. Hence the quantities (11.2) can vanish if, and only if, the corresponding determinant minors of $\| A^{\alpha u} \|$ vanish. Therefore if $\mathbf{A}$ is of simple rank r on the m-th index it follows that

$$(\mathbf{A}')^{1,p-1}_{r+1} = 0, \qquad (\mathbf{A}')^{1,p-1}_r \neq 0.$$

Conversely, if these relations hold the coefficients (11.2) with $m = r + 1$ must all vanish and if $m = r$ at least one of them does not vanish. Hence $\| A^{\alpha u} \|$ is of rank r.

THEOREM 13. *The necessary and sufficient condition that a polyadic* $\mathbf{A}^{(p)}$ *be of p-way rank r is that*

$$(\mathbf{A})^p_{r+1} = 0, \qquad (\mathbf{A})^p_r \neq 0.$$

PROOF. If we write

$$\mathbf{A} = A^{\alpha_1 \cdots \alpha_p} \mathbf{a}^1_{\alpha_1} \cdots \mathbf{a}^p_{\alpha_p}, \qquad (\alpha_i = 1, \cdots, n),$$

where the $\mathbf{a}^j_{\alpha_j}$ are any p sets of independent vectors, and apply (8.3) and (8.2) with $c = 1$, $t = p$ we find that

$$(\mathbf{A})^p_m = \frac{1}{m!} \mathbf{A}^*_p \mathbf{A}^\circ \mathbf{A}^{m-2}, \qquad (m \leqq n^{cj}; j = 1, \cdots, t),$$

$$= \frac{1}{m!} A^{\alpha_{11} \cdots \alpha_{1p}} \cdots A^{\alpha_{m1} \cdots \alpha_{mp}} ((\pm 1)^{p-1} (\mathbf{a}^1_{\alpha_{11}} \cdots \mathbf{a}_{\alpha_{m1}})) ((\mathbf{a}^2_{\alpha_{12}} \cdots \mathbf{a}^2_{\alpha_{m2}})) \cdots ((\mathbf{a}^p_{\alpha_{1p}} \cdots \mathbf{a}^p_{\alpha_{mp}})).$$

Or if we place $\mathbf{a}^\gamma_{\alpha ij} = a^{\gamma,\beta}_{\alpha ij} \mathbf{e}_\beta$ we have upon applying (5.7) to each of the mixed products

$$(\mathbf{A})^p_m = \frac{1}{m!} (\epsilon_{\beta_{11} \cdots \beta_{n1}})^{p-1} \epsilon_{\beta_{12} \cdots \beta_{n2}} \cdots \epsilon_{\beta_{1p} \cdots \beta_{np}} a^{1,\beta_{11}}_{\alpha_{11}} \cdots a^{1,\beta_{m1}}_{\alpha_{m1}} \cdots a^{p,\beta_{1p}}_{\alpha_{1p}} \cdots a^{p,\beta_{mp}}_{\alpha_{mp}} A^{\alpha_{11} \cdots \alpha_{1p}} \cdots A^{\alpha_{m1} \cdots \alpha_{mp}} \mathbf{e}^{\beta_{s1}} \cdots \mathbf{e}^{\beta_{n1}} \cdots \mathbf{e}^{\beta_{np}} \cdots \mathbf{e}^{\beta_{sp}}, \tag{11.3}$$

$$(s = m + 1).$$

Consider the coefficients of the polyads $\mathbf{e}^{\beta_{s1}} \cdots \mathbf{e}^{\beta_{sp}}$ in the right member above, namely

$$\frac{1}{m!} (\epsilon_{\beta_{11} \cdots \beta_{n1}})^{p-1} \epsilon_{\beta_{12} \cdots \beta_{n2}} \cdots \epsilon_{\beta_{1p} \cdots \beta_{np}} M^{\beta_{11} \cdots \beta_{1p}} \cdots M^{\beta_{m1} \cdots \beta_{mp}}, \tag{11.4}$$

where we have made the substitution

$$a^{1,\beta_{j1}}_{\alpha_{j1}} \cdots a^{p,\beta_{jp}}_{\alpha_{jp}} A^{\alpha_{j1} \cdots \alpha_{jp}} = M^{\beta_{j1} \cdots \beta_{jp}}.$$

For each set of values of the $p(n-m)$ free indices in (11.4) the resulting quantity is the p-way m-rowed determinant minor (full signed if p is even, first index non-signant if p is odd) of the p-way matrix $\| M^{\beta_1 \cdots \beta_p} \|$ formed by striking out the layers of this matrix corresponding to the selected set of numbers for the free indices.

Owing to the independence of the polyads $\mathbf{e}^{\beta_{s1}} \cdots \mathbf{e}^{\beta_{sp}}$, $(\mathbf{A})_m^p$ vanishes if, and only if, all the coefficients (11.4) vanish. That is, if $\| M^{\beta_1 \cdots \beta_p} \|$ is of p-way rank r then

$$(\mathbf{A})_{r+1}^p = 0, \qquad (\mathbf{A})^p \neq 0,$$

and conversely.

But $\| M^{\beta_1 \cdots \beta_p} \|$ is the Cayley product of the matrix $\| A^{\alpha_1 \cdots \alpha_p} \|$ with the two-way matrices $\| a_\alpha^{1,\beta} \|, \cdots, \| a_\alpha^{p,\beta} \|$ and since the rank of the latter matrices is n, $\| M^{\beta_1 \cdots \beta_p} \|$ is of the same rank as $\| A^{\alpha_1 \cdots \alpha_p} \|$.

We have the following theorem which is a generalization of the Theorems 12 and 13.

THEOREM 14. *The necessary and sufficient condition that the t-way multipartite rank of the matrix $\| A^{u_1 \cdots u_t} \|$ of a polyadic $\mathbf{A}$ be r is that*

$$(\mathbf{A})_{r+1}^{c_1 \cdots c_t} = 0, \qquad (\mathbf{A})_r^{c_1 \cdots c_t} \neq 0,$$

where c_i is the number of indices included in the partition u_i.

The proof follows very closely that of Theorem 13. With the aid of (2.3) $\mathbf{A}$ can be expressed in terms of t sets of basis polyadics $\mathbf{B}_u^j$, $(j = 1, \cdots, t)$.

$$\mathbf{A} = A^{w_1 \cdots w_t} \mathbf{B}_{w_1}^1 \cdots \mathbf{B}_{w_t}^t, \qquad (w_i = 1, \cdots, k_i = n^{c_i}).$$

Corresponding to equations (11.3) we obtain

$$(\mathbf{A})_m^{c_1 \cdots c_t} = \frac{1}{m!} (\epsilon_{u_{11} \cdots u_{k1}})^{t-1} \epsilon_{u_{12} \cdots u_{g2}} \cdots \epsilon_{u_{1t} \cdots u_{kt}} B_{w_{11}}^{1,u_{11}} \cdots B_{w_{m1}}^{1,u_{m1}} \cdots B_{w_{1t}}^{t,u_{1t}}$$
$$\cdots B_{w_{mt}}^{t,u_{mt}} A^{w_{11} \cdots w_{1t}} \cdots A^{w_{m1} \cdots w_{mt}} \mathbf{E}_{(c_1)}^{u_{s1}} \cdots \mathbf{E}_{(c_1)}^{u_{k1}} \cdots \mathbf{E}_{(c_t)}^{u_{st}} \cdots \mathbf{E}_{(c_t)}^{u_{kt}},$$
$$(k = k_1; g = k_2; h = k_t; s = m + 1; u_{ij} = 1, \cdots, k_j = n^{c_j}),$$

in which the scalar coefficients are

$$\frac{1}{m!} (\epsilon_{u_{11} \cdots u_{k1}})^{t-1} \epsilon_{u_{12} \cdots u_{g2}} \cdots \epsilon_{u_{1t} \cdots u_{kt}} L^{u_{11} \cdots u_{1t}} \cdots L^{u_{m1} \cdots u_{mt}},$$

where

$$L^{u_{j1} \cdots u_{jt}} = B_{w_{j1}}^{1,u_{j1}} \cdots B_{w_{jt}}^{t,u_{jt}} A^{w_{j1} \cdots w_{jt}}, \qquad (j = 1, \cdots, m).$$

The proof then follows as in Theorem 13.

It is to be noted that in the proofs of Theorems 12, 13, and 14 it was shown that the various ranks of $\mathbf{A}$ depended solely upon the vanishing or non-vanishing of the functions $(\mathbf{A})_m^{c_1 \cdots c_t}$, and were independent of the basis polyadics chosen for the representation of $\mathbf{A}$. We have therefore established

THEOREM 15. *If the indices of the components* $A^{a_1 \cdots a_p}$ *of a polyadic* **A** *are divided in any way into t sets* $u_1, \cdots, u_t$, $(t = 2, \cdots, p)$, *the t-way rank (simple or multipartite) of the matrix* $\| A^{u_1 \cdots u_t} \|$ *is invariant under any non-singular transformation on any index or set of indices of* $A^{a_1 \cdots a_t}$.

Or stated otherwise.

The t-way rank, $(t = 2, \cdots, p)$, *simple or multipartite, of a polyadic is invariant under any transformation on any vector file, or any set of vector files, provided the mixed product of the polyadics involved in the transformation does not vanish.*

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COEFFICIENTS FOR NUMERICAL DIFFERENTIATION WITH CENTRAL DIFFERENCES

By Herbert E. Salzer*

The following table lists the coefficients $A^n_{2\nu-1}$ and $A^n_{2\nu}$ of mean central and central differences respectively, in the formula for the nth derivative in terms of central differences, namely,

$$\omega^n \phi_0^{(n)} = \sum_{\nu=\frac{n+1}{2}}^{\nu=k} A^n_{2\nu-1} \square \delta_0^{2\nu-1} + R_1 \text{ for } n \text{ odd, and}$$

$$\omega^n \phi_0^{(n)} = \sum_{\nu=\frac{n}{2}}^{\nu=k} A^n_{2\nu} \delta_0^{2\nu} + R_2 \text{ for } n \text{ even.}^1$$

Here ϕ_0 is an abbreviation for $\phi(u_0)$, the value of $\phi(u)$ at some interval point, ω is the tabular interval and $\square$ is the operator $\dfrac{E^{\frac{1}{2}} + E^{-\frac{1}{2}}}{2}$.

Due to the fundamental importance of these coefficients in applied and pure mathematics,[2] because of their greater convenience when compared with coefficients of advancing differences, and because the exact forms of R_1 and R_2 involve coefficients of order considerably greater than that in the last coefficient used, it was the author's purpose to make this table the "ultimate" in coefficients for numerical differentiation. In brief, coefficients are given for derivatives as far as the 52nd, most of them going as far as the 42nd difference, and the rest going as far as some difference between the 42nd and 52nd. For the first 30 derivatives, exact values are given for coefficients of the first 30 differences, and also exact values are given for some coefficients of differences beyond the 30th. For all derivatives beyond the 30th, exact values are given for coefficients of differences going as far as some difference between the 41st and 52nd. Elsewhere, that is, for most of the coefficients of the 31st to 42nd differences, 18 significant figures are given, with accuracy to within 0.6 unit in the last significant figure. Wherever exact values are given, it is believed that the fractions are in lowest terms. Since the denominators are highly composite numbers whose factors consist of powers of the smallest primes, these fractions can readily be converted into decimal form with the aid of a 10- or 8-bank computing machine.

To obtain a formula somewhat similar to the Everett interpolation formula

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[1] For the forms of R_1 and R_2, see Milne-Thomson, *Calculus of Finite Differences*, p. 160, and Steffensen, *Interpolation*, pp. 66–67.

[2] For n even, $A^n_{2\nu} = [n/2\nu\ (2\nu - n)!]\ B^{\{2\nu\}}_{2\nu-n}(\nu)$ and for n odd $A^n_{2\nu-1} = [1/(2\nu - n - 1)!]$. $B^{\{2\nu\}}_{2\nu-n-1}(\nu)$ where $B^{\{2\nu\}}_{2\nu-n}(\nu)$ and $B^{\{2\nu\}}_{2\nu-n-1}(\nu)$ denote the $(2\nu - n)$th and $(2\nu - n - 1)$th Bernoulli polynomials of the (2ν)th order, for $x = \nu$. (See Milne-Thomson, pp. 127, 160.)

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which employs central differences of only even order, it suffices merely to write $\frac{1}{2}(\delta_1^{2\nu-2} - \delta_{-1}^{2\nu-2})$ for $\square\delta_0^{2\nu-1}$ and thus, for n odd,

$$\omega^n \phi_0^{(n)} = \sum_{\nu=\frac{n+1}{2}}^{\nu=k} (\tfrac{1}{2}A_{2\nu-1}^n \delta_1^{2\nu-2} - \tfrac{1}{2}A_{2\nu-1}^n \delta_{-1}^{2\nu-2}) + R_1$$

The chief use of these coefficients in computational work is to find the successive derivatives of functions that have already been tabulated to a sufficiently large number of decimal places at some suitable interval. Many calculated expressions can be made to difference well by use of extra factors and terms that make formal differentiation unfeasible or even impossible. Then numerical differentiation by means of these coefficients is the most practical procedure. For any polynomial (or function calculated from a polynomial) these coefficients give exact values for the derivatives of that polynomial (exclusive of rounding errors). Furthermore these coefficients can be employed whenever it is desired to extend the calculation of an analytic function into a neighboring two dimensional region when it has already been tabulated at regular intervals along a straight line in the region. The advantage of the large number of coefficients comes in when certain functions, such as Bessel, Legendre, error, gamma, etc., require a difference polynomial of very high degree to approximate them within a region. Since differences of the nth order are inaccurate to $2^n \times$ error in tabulated function, and also since ω^n is small for $\omega < 1$ and n large, the calculation of many functions to a number of decimal places far beyond their immediate use, is seen to be vital to their use in numerical differentiation and extension into the complex plane. Another possible use of the extremely large number of coefficients might occur when we are interested in an upper bound for the error in using the formula with a fixed number of differences. Then, if we can set an upper bound for the nth derivative of the function, the error is majorized by a known expression containing central difference coefficients of order much greater than that of the last difference employed. (See footnote 1.)

The coefficients were calculated by first obtaining the central factorial polynomials $x^{[2\nu]}$ and dividing their coefficients by the proper products of consecutive integers in accordance with (2) and (3) (See below). The polynomials $x^{[2\nu]}$ were checked by the fact that the sums of their coefficients were 0. The divisions were checked by multiplications and wherever the division involved exact cancellation, the checkback involved two exact multiplications.

In addition, all these values were checked to about 10 significant figures by a recurrence formula which was found for these coefficients. Since this formula appears to be new, its derivation is given below in detail. The formula can also be obtained from the definition of A_m^n (see footnote 2.) and the recurrence formulae for Bernoulli polynomials; but the derivation by that means is longer and more indirect than the following:

Let $\phi(u_0 + \omega x)$ be written as $\phi_{\omega x}$ or f_x, where x is the tabular interval. Then $\omega^n \phi^{(n)}(u_0 + \omega x) = \omega^n \phi_{\omega x}^{(n)} = f_x^{(n)}; \omega^n \phi_0^{(n)} = f_0^{(n)}, \phi_0 = f_0$. By Stirling's interpolation formula,

$$(1) \qquad f_x = \sum_{\nu=0}^{\nu=k} \left(\frac{x^{[2\nu]-1}}{(2\nu-1)!} \square\delta_0^{2\nu-1} + \frac{x^{[2\nu]}}{(2\nu)!} \delta_0^{2\nu} \right) + \frac{x^{[2k+2]-1}}{(2k+1)!} f_\theta^{(2k+1)},$$

where $0 \leq \theta \leq 1$ and θ depends upon x. (See Steffensen, p. 27.) Here $x^{[2\nu]} \equiv x^2(x^2 - 1^2)(x^2 - 2^2) \cdots (x^2 - \overline{\nu - 1}^2)$, $x^{[2\nu]-1} \equiv x^{[2\nu]}/x$. Let C_m^n denote the coefficient of x^m in $x(x^2 + 1^2)(x^2 + 2^2) \cdots (x^2 + n^2)$. Differentiating n times and letting $x = 0$, we obtain:
For n odd,

$$(2) \qquad f_0^{(n)} = \omega^n \phi_0^{(n)} = \sum_{\nu=0}^{\nu=k} (-1)^{\nu-\frac{n+1}{2}} \frac{n! C_n^{\nu-1}}{(2\nu-1)!} \square \delta_0^{2\nu-1} + R_1$$

$$= \sum_{\nu=0}^{\nu=k} A_{2\nu-1}^n \square \delta_0^{2\nu-1} + R_1$$

For n even,

$$(3) \qquad f_0^{(n)} = \omega^n \phi_0^{(n)} = \sum_{\nu=0}^{\nu=k} (-1)^{\nu-\frac{n}{2}} \frac{n!}{(2\nu)!} C_{n-1}^{\nu-1} \delta_0^{2\nu} + R_2$$

$$= \sum_{\nu=0}^{\nu=k} A_{2\nu}^n \delta_0^{2\nu} + R_2$$

(See footnotes 1. and 2.)
Effectively the summation begins with $\nu = \frac{n+1}{2}$ for (2) and $\nu = \frac{n}{2}$ for (3).
Obviously, from (2) and (3),

$$(4) \qquad A_{2\nu}^{n+1} = \frac{n+1}{2\nu} \cdot A_{2\nu-1}^n \text{ if } n \text{ is odd.}$$

From the evident relationship

$$(5) \qquad C_n^{\nu-1} = C_{n-2}^{\nu-2} + (\nu-1)^2 C_n^{\nu-2},$$

for $\nu > 2$, after multiplication of both members by $(-1)^{\nu-\frac{n+1}{2}} n!/(2\nu-1)!$ it is seen from (2) that for odd n the left member is now $A_{2\nu-1}^n$. The first term on the right is

$$\frac{n}{2\nu-1} \frac{(n-1)!}{(2\nu-2)!} C_{n-2}^{\nu-2} (-1)^{\nu-\frac{n}{2}-\frac{1}{2}}$$

But from (3),

$$A_{2\nu-2}^{n-1} = C_{n-2}^{\nu-2} \frac{(n-1)!}{(2\nu-2)!} (-1)^{\nu-1-\frac{n-1}{2}},$$

making the term equal to

$$\frac{n}{2\nu-1} A_{2\nu-2}^{n-1}.$$

The second term on the right can be written as

$$(-1)^1(\nu-1)^2 \frac{1}{n+1} \cdot \frac{1}{2\nu-1} \cdot \frac{(n+1)!}{(2\nu-2)!} C_n^{\nu-2} (-1)^{\nu-1-\frac{n+1}{2}}.$$

But from (3),

$$A_{2\nu-2}^{n+1} = \frac{(n+1)!}{(2\nu-2)!} C_n^{\nu-2} (-1)^{\nu-1-\frac{n+1}{2}};$$

hence the term is

$$-(\nu-1)^2 \cdot \frac{1}{2\nu-1} \cdot \frac{1}{n+1} A_{2\nu-2}^{n+1}.$$

Thus the final recursion formula is

$$A_{2\nu-1}^{n} = \frac{1}{2\nu-1}\left[nA_{2\nu-2}^{n-1} - (\nu-1)^2 \cdot \frac{1}{n+1} \cdot A_{2\nu-2}^{n+1}\right] \tag{6}$$

which is applied only when n is odd, since it is trivially true for n even. Although the foregoing proof is only for $\nu > 2$, formula (6) can be verified also for $\nu = 2$ and $\nu = 1$ provided we adopt the natural notation $A_k^0 = 0$ and $A_0^0 = 1$ respectively. Also for $n = 1$, $A_k^0 = 0$ gives the correct equation for (6) because then in (5) $C_{-1}^{\nu-2} \equiv 0$.

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BIBLIOGRAPHY

J. F. Steffensen. *Interpolation*. Baltimore, Williams & Wilkins Co., 1927, pp. 60–71.

L. M. Milne-Thomson. *Calculus of Finite Differences*. London, Macmillan, 1933, pp. 124–142, 154–162.

H. T. Davis. *Tables of the Higher Mathematical Functions*. Principia Press, Bloomington, Indiana, 1933, vol. 1, pp. 73–78.

E. T. Whittaker and G. Robinson. *Calculus of Observations*. London and Glasgow, Blackie & Son Ltd., 1937, pp. 62–65.

W. G. Bickley and J. C. P. Miller. *Numerical Differentiation near the Limits of a Difference Table*. Philosophical Magazine, (7), vol. 33, Jan. 1942, pp. 1–14.

COEFFICIENTS OF $\square\delta^n$ FOR n ODD AND OF δ^n FOR n EVEN

$n =$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
D1	1		$-\frac{1}{6}$		$\frac{1}{30}$		$-\frac{1}{140}$		$\frac{1}{630}$		$-\frac{1}{2772}$		$\frac{1}{12012}$		$-\frac{1}{51480}$		$\frac{1}{218790}$		$-\frac{1}{923780}$	
D2		1		$-\frac{1}{12}$		$\frac{1}{90}$		$-\frac{1}{560}$		$\frac{1}{3150}$		$-\frac{1}{16632}$		$\frac{1}{84084}$		$-\frac{1}{411840}$		$\frac{1}{1969110}$		$-\frac{1}{9237800}$
D3			1		$-\frac{1}{4}$		$\frac{7}{120}$		$-\frac{41}{3024}$		$\frac{479}{151200}$		$-\frac{59}{79200}$		$\frac{266681}{1513512000}$		$-\frac{63397}{1513512000}$		$\frac{514639}{51459408000}$	
D4				1		$-\frac{1}{6}$		$\frac{7}{240}$		$-\frac{41}{7560}$		$\frac{479}{453600}$		$-\frac{59}{277200}$		$\frac{266681}{6054048000}$		$-\frac{63397}{6810804000}$		$\frac{514639}{257297040000}$
D5					1		$-\frac{1}{3}$		$\frac{13}{144}$		$-\frac{139}{6048}$		$\frac{37}{6480}$		$-\frac{4201}{2993760}$		$\frac{3739217}{10897286400}$		$-\frac{364919}{4358914560}$	
D6						1		$-\frac{1}{4}$		$\frac{13}{240}$		$-\frac{139}{12096}$		$\frac{37}{15120}$		$-\frac{4201}{7983360}$		$\frac{3739217}{32691859200}$		$-\frac{364919}{14529715200}$
D7							1		$-\frac{5}{12}$		$\frac{31}{240}$		$-\frac{311}{8640}$		$\frac{2473}{259200}$		$-\frac{4679}{1900800}$		$\frac{5839219}{9340531200}$	
D8								1		$-\frac{1}{3}$		$\frac{31}{360}$		$-\frac{311}{15120}$		$\frac{2473}{518400}$		$-\frac{4679}{4276800}$		$\frac{5839219}{23351328000}$
D9									1		$-\frac{1}{2}$		$\frac{7}{40}$		$-\frac{67}{1260}$		$\frac{2021}{134400}$		$-\frac{21713}{5322240}$	
D10										1		$-\frac{5}{12}$		$\frac{1}{8}$		$-\frac{67}{2016}$		$\frac{2021}{241920}$		$-\frac{21713}{10644480}$
D11											1		$-\frac{7}{12}$		$\frac{41}{180}$		$-\frac{757}{10080}$		$\frac{5473}{241920}$	
D12												1		$-\frac{1}{2}$		$\frac{41}{240}$		$-\frac{757}{15120}$		$\frac{5473}{403200}$

D13													1		$-\frac{2}{3}$		$\frac{23}{90}$		$-\frac{619}{6048}$	
D14														1		$-\frac{7}{12}$		$\frac{161}{720}$		$-\frac{619}{8640}$
D15															1		$-\frac{3}{4}$		$\frac{17}{48}$	
D16																1		$-\frac{2}{3}$		$\frac{17}{60}$
D17																	1		$-\frac{5}{6}$	
D18																		1		$-\frac{3}{4}$
D19																			1	
D20																				1

COEFFICIENTS OF $\square\delta^n$ FOR n ODD AND OF δ^n FOR n EVEN—*Continued*

$n =$	21	22	23	24	25	26	27	28	29	30
D1	$\frac{1}{3879876}$		$-\frac{1}{16224936}$		$\frac{1}{67603900}$		$-\frac{1}{280816200}$		$\frac{1}{1163381400}$	
D2		$\frac{1}{42078636}$		$-\frac{1}{194699232}$		$\frac{1}{878850700}$		$-\frac{1}{3931426800}$		$\frac{1}{17450721000}$
D3	$-\frac{178939}{74662922880}$		$\frac{10410343}{19068427336960}$		$-\frac{18500393}{13319673984000}$		$\frac{40799043101}{12155534490939840000}$		$-\frac{1411432849}{173650492727712000}$	
D4		$-\frac{178939}{410646075840}$		$\frac{10410343}{108410564021760}$		$-\frac{18500393}{865778809896000}$		$\frac{40799043101}{8506874143657889000}$		$-\frac{1411432849}{1302378695457840000}$
D5	$\frac{1473061}{72176832000}$		$-\frac{7354899857}{1478325873024000}$		$\frac{1053912709}{86867439120000}$		$-\frac{593151411269}{2003659531473600000}$		$\frac{92257290523249}{12763311215486832000000}$	
D6		$\frac{1473061}{264648384000}$		$-\frac{7354899857}{5913303492096000}$		$\frac{1053912709}{3764255695200000}$		$-\frac{593151411269}{9350411146876800000}$		$\frac{92257290523249}{63816558077434160000000}$
D7	$-\frac{800993}{5094835200}$		$\frac{374322799}{9527341824000}$		$-\frac{211789363}{21660452352000}$		$\frac{1449491072231}{59736433236480000}$		$-\frac{35791253123}{5955149071872000}$	
D8		$-\frac{800993}{14010796800}$		$\frac{374322799}{28582025472000}$		$-\frac{211789363}{70396470144000}$		$\frac{1449491072231}{209077516327680000}$		$-\frac{35791253123}{2233180901952000000}$
D9	$\frac{10684483}{9906624000}$		$-\frac{61085281}{21794572800}$		$\frac{615453101}{8850178560000}$		$-\frac{2090586163331}{114042281633280000}$		$\frac{909495616199}{196010171557200000}$	
D10		$\frac{10684483}{21794572800}$		$-\frac{61085281}{523069747200}$		$\frac{615453101}{22230464256000}$		$-\frac{2090586163331}{319318388573184000}$		$\frac{909495616199}{58803051467160000}$
D11	$-\frac{2677}{414720}$		$\frac{14091479}{7925299200}$		$-\frac{1748857}{3657830400}$		$\frac{437843543}{346448793600}$		$-\frac{95763529279}{2902894441574400}$	

D12		$-\frac{2677}{700320}$		$\frac{14091479}{15850598400}$		$-\frac{1748857}{7925299200}$		$\frac{437843543}{8083805184000}$		$-\frac{95763529279}{7257236103936000}$
D13	$\frac{118921}{3628800}$		$-\frac{392251}{39916800}$		$\frac{189156857}{67060224000}$		$-\frac{45139487}{57480192000}$		$\frac{365667859}{1710035712000}$	
D14		$\frac{10811}{518400}$		$-\frac{392251}{68428800}$		$\frac{189156857}{124540416000}$		$-\frac{45139487}{114960384000}$		$\frac{365667859}{3664362240000}$
D15	$-\frac{1639}{12096}$		$\frac{11129}{241920}$		$-\frac{462397}{31933440}$		$\frac{323535929}{74724249600}$		$-\frac{74616869}{59779399680}$	
D16		$-\frac{149}{1512}$		$\frac{11129}{362880}$		$-\frac{35569}{3991680}$		$\frac{323535929}{130767436800}$		$-\frac{74616869}{112086374400}$
D17	$\frac{77}{180}$		$-\frac{331}{1890}$		$\frac{114101}{1814400}$		$-\frac{354971}{17107200}$		$\frac{844617209}{130767436800}$	
D18		$\frac{7}{20}$		$-\frac{331}{2520}$		$\frac{8777}{201600}$		$-\frac{354971}{26611200}$		$\frac{844617209}{217945728000}$
D19	$-\frac{11}{12}$		$\frac{61}{120}$		$-\frac{559}{2520}$		$\frac{7261}{86400}$		$-\frac{154477}{5322240}$	
D20		$-\frac{5}{6}$		$\frac{61}{144}$		$-\frac{43}{252}$		$\frac{7261}{120960}$		$-\frac{154477}{7983360}$
D21	1		-1		$\frac{143}{240}$		$-\frac{1193}{4320}$		$\frac{1903}{17280}$	
D22		1		$-\frac{11}{12}$		$\frac{121}{240}$		$-\frac{13123}{60480}$		$\frac{20033}{259200}$
D23			1		$\frac{13}{12}$		$\frac{497}{720}$		$-\frac{4097}{13096}$	
D24				1		-1		$\frac{71}{120}$		$-\frac{4097}{15120}$

COEFFICIENTS OF $\square\delta^n$ FOR n ODD AND OF δ^n FOR n EVEN—*Continued*

$n =$	21	22	23	24	25	26	27	28	29	30
D25					1		$-\frac{7}{6}$		$\frac{19}{24}$	
D26						1		$-\frac{13}{12}$		$\frac{247}{360}$
D27							1		$-\frac{5}{4}$	
D28								1		$-\frac{7}{6}$
D29									1	
D30										1

COEFFICIENTS OF $\square\delta^n$ FOR n ODD AND OF δ^n FOR n EVEN—*Continued*

$n =$	31	32	33	34
D1	−.(9) 207958872190124186		.(10) 504142720460907118	
D2		−.(10) 129074295118827616		.(11) 296554541447592422
D3	.(8) 197199947345435825		−.(9) 479243062914258009	
D4		.(9) 246499934181794782		−.(10) 563814191663832952
D5	−.(7) 17662665505127041		.(8) 431920677914820020	
D6		−.(8) 331174978221050703		.(9) 762212961026152977
D7	.(6) 148670907488906607		−.(7) 367439244965674668	
D8		.(7) 371677268722266517		−.(8) 864562929330999218
D9	−.(5) 116912111983929049		.(6) 293559954562556477	
D10		−.(6) 365350349949778277		.(7) 863411631066342581
D11	.(5) 853002734105294141		−.(5) 218966886599609371	
D12		.(5) 319876025289485303		−.(6) 772824305645680132
D13	−.(4) 572682453630974676		.(4) 151433286724973565	
D14		−.(4) 250548573463551421		.(5) 623548827691067703
D15	.(3) 350270406592425417		−.(4) 963026094828706203	
D16		.(3) 175135203296212709		−.(4) 453188780507626448
D17	−.(2) 19277094423148215O		.(3) 55754466501642063	
D18		−.(2) 10843365613020871O		.(3) 295170705794986974
D19	.(2) 939734921830556451		−.(2) 290246025415709609	

D20		.(2) 587334329965397782		−.(2) 170732956136888005

D20		.(2) 587334325965597782		−.(2) 170732956126888005
D21	$-\frac{1889}{47520}$		.(1) 133743538322531378	
D22		$-\frac{1889}{69120}$		.(2) 865399365616379505
D23	$\frac{128699}{907200}$		$-\frac{12798671}{239500800}$	
D24		$\frac{128699}{1209600}$		$-\frac{752863}{19958400}$
D25	$-\frac{155}{378}$		$\frac{26129}{145152}$	
D26		$-\frac{2015}{6048}$		$\frac{19981}{145152}$
D27	$\frac{9}{10}$		$-\frac{1649}{3360}$	
D28		$\frac{63}{80}$		$-\frac{97}{240}$
D29	$-\frac{4}{3}$		$\frac{731}{720}$	
D30		$-\frac{5}{4}$		$\frac{43}{48}$
D31	1		$-\frac{17}{12}$	
D32		1		$-\frac{4}{3}$
D33			1	
D34				1

NOTE: The numbers in parentheses indicate the number of zeros following the decimal point.

COEFFICIENTS OF $\square\delta^n$ FOR n ODD AND OF δ^n FOR n EVEN—*Continued*

$n =$	35	36	37	38
D1	−.(10) 122434600683363157		.(11) 297814039500072545	
D2		−.(12) 690192559352017540		.(12) 1567442313158327655
D3	.(9) 116641547743274881		−.(10) 284274191180413510	
D4		.(10) 129001719714749868		−.(11) 2992359907162347 47
D5	−.(8) 105700470624547481		.(9) 258860627461200762	
D6		−.(9) 176167451040912468		.(10) 408727306517780150
D7	.(8) 90759671129001 8681		−.(8) 224099665331048834	
D8		.(8) 201688158062226374		−.(9) 471788769117997545
D9	−.(7) 735162936406148568		.(7) 183729345132738942	
D10		−.(7) 204211926779485713		.(8) 433498270665102478
D11	.(6) 558912518718279239		−.(6) 142022859133025606	
D12		.(6) 186304172906093080		−.(7) 448493230367449283
D13	−.(5) 396471456350061111		.(5) 102984825164391108	
D14		−.(5) 154183344099579321		.(6) 379417770921440925
D15	.(4) 260601287073731551		−.(5) 696401783733230524	
D16		.(4) 115822794254991800		−.(5) 293221803677149604
D17	−.(3) 157415729746483500		.(4) 436119004851847980	
D18		−.(4) 787078648732417500		.(4) 2065826865087701 00
D19	.(3) 865118730584001979		−.(3) 250851838049935434	
D20		.(3) 480621516991112211		−.(3) 1320272831841765 44
D21	−.(2) 427245500602280436		.(2) 131203101949696907	
D22		−.(2) 261094478000282488		.(3) 7595969006187831 46

D23	.(1) 186649405511632094		−.(2) 616314040327660533	
D24		.(1) 124432937007755129		−.(2) 389250972838522442
D25	$-\frac{10745}{152064}$		.(1) 255954508010972953	
D26		$-\frac{139685}{2737152}$		.(1) 175120766639086761
D27	$\frac{10097}{44800}$		$-\frac{1633259}{17740800}$	
D28		$\frac{10097}{87000}$		$-\frac{85961}{1267200}$
D29	$-\frac{5861}{10080}$		$\frac{112423}{403200}$	
D30		$-\frac{5861}{12096}$		$\frac{5917}{26880}$
D31	$\frac{91}{80}$		$-\frac{41287}{60480}$	
D32		$\frac{91}{90}$		$-\frac{2173}{3780}$
D33	$-\frac{3}{2}$		$\frac{19}{15}$	
D34		$-\frac{17}{12}$		$\frac{17}{15}$
D35	1		$-\frac{19}{12}$	
D36		1		$-\frac{3}{2}$
D37			1	
D38				1

COEFFICIENTS OF $\square\delta^n$ FOR n ODD AND OF δ^n FOR n EVEN—*Continued*

$n =$	39	40	41	42
D1	−.(12) 725444455192484404		.(12) 176937671998166928	
D2		−.(13) 362722227596242202		.(14) 842560342848413942
D3	.(11) 693668498244462352		−.(11) 169452845104207580	
D4		.(12) 693668498244462352		−.(12) 161383662004007219
D5	−.(10) 634394297543022686		.(10) 155576253666889024	
D6		−.(11) 951591446314534029		.(11) 222251790952698605
D7	.(9) 553219931308002188		−.(9) 136556358885903397	
D8		.(9) 110643986261600438		−.(10) 260107350258863613
D9	−.(8) 458433273841446084		.(8) 114241764049997591	
D10		−.(8) 114608318460361521		.(9) 272004200119041883
D11	.(7) 359590249281257572		−.(8) 907797961736334975	
D12		.(7) 107877074784377272		−.(8) 259370846210381421
D13	−.(6) 265810246199867676		.(7) 682522599809357856	
D14		−.(7) 930335861699536865		.(7) 227507533269785952
D15	.(5) 184229323355073112		−.(6) 483376490928210158	
D16		.(6) 736917293420292448		−.(6) 184143425115508632

D17	−.(4) 11901557980368488 1		.(5) 32083700924836549 5	
D18		−.(5) 53557010911658196 7		.(5) 13750157539215664 1
D19	.(4) 71169219636949995 2		−.(4) 19840256514674403 1	
D20		.(4) 35584609818474997 6		−.(5) 94477411974640014 6
D21	−.(3) 39068891364087445 1		.(3) 11351624250235901 1	
D22		−.(3) 21487890250248094 8		.(4) 59460888929807101 2
D23	.(2) 19492451966553785 4		−.(3) 59596748107587432 2	
D24		.(2) 11695471179932271 2		−.(3) 34055284632907104 1
D25	−.(2) 87299878415398697 6		.(2) 28424038137860823 3	
D26		−.(2) 56744920970009153 4		.(2) 17595833132961462 1
D27	.(1) 34549594991373265 2		−.(1) 12163591134945301 6	
D28		.(1) 24184716493961285 6		−.(2) 81090607566302010 7
D29	$-\frac{34997}{295680}$		.(1) 45974815573122864 8	
D30		$-\frac{34997}{394240}$		.(1) 32839153980802046 3
D31	$\frac{247673}{725760}$		$-\frac{685901}{4561920}$	

COEFFICIENTS OF $\square\delta^n$ FOR n ODD AND OF δ^n FOR n EVEN—*Continued*

	39	40	41	42
D32		$\frac{247673}{907200}$		$-\frac{685901}{5987520}$
D33	$-\frac{601}{756}$		$\frac{17869}{43200}$	
D34		$-\frac{10217}{15120}$		$\frac{303773}{907200}$
D35	$\frac{101}{72}$		$-\frac{397}{432}$	
D36		$\frac{101}{80}$		$-\frac{397}{504}$
D37	$-\frac{5}{3}$		$\frac{371}{240}$	
D38		$-\frac{19}{12}$		$\frac{1007}{720}$
D39	1		$-\frac{7}{4}$	
D40		1		$-\frac{5}{3}$
D41			1	
D42				1

COEFFICIENTS OF $\square\delta^n$ FOR n ODD AND OF δ^n FOR n EVEN—*Continued*

$n =$	43	44	45	46	47	48	49	50	51	52
D33	$-\frac{342773}{1814400}$									
D34		$-\frac{5827141}{39916800}$								
D35	$\frac{12881}{25920}$		$-\frac{804103}{3421440}$							
D36		$\frac{1171}{2880}$		$-\frac{34961}{190080}$						
D37	$-\frac{10637}{10080}$		$\frac{179101}{302400}$		$-\frac{240923}{831600}$					
D38		$-\frac{18373}{20160}$		$\frac{147953}{302400}$		$-\frac{4577537}{19958400}$				
D39	$\frac{407}{240}$		$-\frac{72841}{60480}$		$\frac{423767}{604800}$		$-\frac{13463}{38016}$			
D40		$\frac{37}{24}$		$-\frac{3167}{3024}$		$\frac{423767}{725760}$		$-\frac{13463}{47520}$		
D41	$-\frac{11}{6}$		$\frac{667}{360}$		$-\frac{5167}{3780}$		$\frac{119507}{145152}$		$-\frac{20575789}{47900160}$	
D42		$-\frac{7}{4}$		$\frac{203}{120}$		$-\frac{5167}{4320}$		$\frac{119507}{172800}$		$-\frac{1582753}{4561920}$

D43	1		$-\frac{23}{12}$		$\frac{121}{60}$		$-\frac{9335}{6048}$		$\frac{3488771}{3628800}$	
D44		1		$-\frac{11}{6}$		$\frac{1331}{720}$		$-\frac{20537}{15120}$		$\frac{2952037}{3628800}$
D45			1		-2		$\frac{35}{16}$		$-\frac{3497}{2016}$	
D46				1		$-\frac{23}{12}$		$\frac{161}{80}$		$-\frac{6187}{4032}$
D47					1		$-\frac{25}{12}$		$\frac{1703}{720}$	
D48						1		-2		$\frac{131}{60}$
D49							1		$-\frac{13}{6}$	
D50								1		$-\frac{25}{12}$
D51									1	
D52										1

TABLE OF FOURIER COEFFICIENTS*

By Arnold N. Lowan and Jack Laderman

The solutions of problems in heat conduction, diffusion, and wave motion for domains bounded by parallel planes involve the Fourier Coefficients

$$\frac{2}{a}\int_0^a f(x)\sin\frac{n\pi x}{a}\,dx \quad \text{and} \quad \frac{2}{a}\int_0^a f(x)\cos\frac{n\pi x}{a}\,dx.$$

These integrals are also encountered in the solution of problems in the potential theory. The substitution of x by ax reduces the two integrals to the form

$$\int_0^1 \varphi(x)\sin n\pi x\,dx; \qquad \int_0^1 \varphi(x)\cos n\pi x\,dx$$

where $\varphi(x)$ is written for $2f(ax)$.

Whenever $\varphi(x)$ is a known polynomial whose degree does not exceed 10, the present table of the functions

$$S(k, n) = \int_0^1 x^k \sin n\pi x\,dx \quad \text{and} \quad C(k, n) = \int_0^1 x^k \cos n\pi x\,dx$$

will facilitate the evaluation of the first hundred Fourier Coefficients.

The tabulated definite integrals satisfy the recurrence formulae:

$$S(k, n) = S(1, n) - \frac{k(k-1)}{n^2\pi^2}\,S(k-2, n)$$

$$C(k, n) = \frac{k}{2}\,C(2, n) - \frac{k(k-1)}{n^2\pi^2}\,C(k-2, n)$$

They also satisfy the "cross relations":

$$S(k, n) = S(1, n) - \frac{k}{n\pi}\,C(k-1, n)$$

$$C(k, n) = -\frac{k}{n\pi}\,S(k-1, n)$$

The first two formulae were used to compute the two integrals in question while the latter two were used in checking the accuracy of the computed values. The computations were performed originally to twenty decimal places with such extreme care, that the "cross relation" checks on these twenty-place values did not reveal a single error. The present table is an abridged version of the original twenty-place manuscript. The reader will note that the plus sign after an entry indicates that the eleventh decimal place is 5 or larger.

* The results reported here were obtained during 1940–41 by the Mathematical Tables Project of the Work Projects Administration operated under the sponsorship of the National Bureau of Standards, Dr. Lyman C. Briggs, Director.

n	$\int_0^1 \sin n\pi x\,dx$	n	$\int_0^1 \sin n\pi x\,dx$	n	$\int_0^1 x \sin n\pi x\,dx$	n	$\int_0^1 x \sin n\pi x\,dx$
1	0.63661 97723+	51	0.01248 27406	1	+0.31830 98861+	51	+0.00624 13703
2	0.00000 00000	52	0.00000 00000	2	−0.15915 49430+	52	−0.00612 13439+
3	0.21220 65907+	53	0.01201 16938	3	+0.10610 32953+	53	+0.00600 58469
4	0.00000 00000	54	0.00000 00000	4	−0.07957 74715	54	−0.00589 46275
5	0.12732 39544+	55	0.01157 49049+	5	+0.06366 19772	55	+0.00578 74524+
6	0.00000 00000	56	0.00000 00000	6	−0.05305 16476+	56	−0.00568 41051
7	0.09094 56817+	57	0.01116 87679	7	+0.04547 28408+	57	+0.00558 43839+
8	0.00000 00000	58	0.00000 00000	8	−0.03978 87357+	58	−0.00548 81014+
9	0.07073 55302+	59	0.01079 01656	9	+0.03536 77651	59	+0.00539 50828
10	0.00000 00000	60	0.00000 00000	10	−0.03183 09886	60	−0.00530 51647+
11	0.05787 45247+	61	0.01043 63897	11	+0.02893 72623+	61	+0.00521 81948+
12	0.00000 00000	62	0.00000 00000	12	−0.02652 58238	62	−0.00513 40304
13	0.04897 07517	63	0.01010 50757+	13	+0.02448 53758+	63	+0.00505 25378+
14	0.00000 00000	64	0.00000 00000	14	−0.02273 64204	64	−0.00497 35919+
15	0.04244 13181+	65	0.00979 41503	15	+0.02122 06590+	65	+0.00489 70751+
16	0.00000 00000	66	0.00000 00000	16	−0.01989 43678+	66	−0.00482 28770+
17	0.03744 82219	67	0.00950 17876	17	+0.01872 41109+	67	+0.00475 08938
18	0.00000 00000	68	0.00000 00000	18	−0.01768 38825+	68	−0.00468 10277
19	0.03350 63038	69	0.00922 63735	19	+0.01675 31519	69	+0.00461 31867+
20	0.00000 00000	70	0.00000 00000	20	−0.01591 54943	70	−0.00454 72840+
21	0.03031 52272+	71	0.00896 64756+	21	+0.01515 76136	71	+0.00448 32378
22	0.00000 00000	72	0.00000 00000	22	−0.01446 86311+	72	−0.00442 09706
23	0.02767 91205	73	0.00872 08187+	23	+0.01383 95602+	73	+0.00436 04093+
24	0.00000 00000	74	0.00000 00000	24	−0.01326 29119	74	−0.00430 14849
25	0.02546 47908+	75	0.00848 82636	25	+0.01273 23954	75	+0.00424 41318
26	0.00000 00000	76	0.00000 00000	26	−0.01224 26879	76	−0.00418 82879+
27	0.02357 85100+	77	0.00826 77892+	27	+0.01178 92550	77	+0.00413 38946
28	0.00000 00000	78	0.00000 00000	28	−0.01136 82102	78	−0.00408 08959+
29	0.02195 24059	79	0.00805 84781	29	+0.01097 62029+	79	+0.00402 92390+
30	0.00000 00000	80	0.00000 00000	30	−0.01061 03295	80	−0.00397 88735+
31	0.02053 61216+	81	0.00785 95033+	31	+0.01026 80608	81	+0.00392 97516+
32	0.00000 00000	82	0.00000 00000	32	−0.00994 71839	82	−0.00388 18278+
33	0.01929 15082+	83	0.00767 01177	33	+0.00964 57541	83	+0.00383 50588+
34	0.00000 00000	84	0.00000 00000	34	−0.00936 20554+	84	−0.00378 94034
35	0.01818 91363+	85	0.00748 96443+	35	+0.00909 45681+	85	+0.00374 48221+
36	0.00000 00000	86	0.00000 00000	36	−0.00884 19412+	86	−0.00370 12777
37	0.01720 59397+	87	0.00731 74686	37	+0.00860 29698+	87	+0.00365 87343
38	0.00000 00000	88	0.00000 00000	38	−0.00837 65759+	88	−0.00361 71577+
39	0.01632 35839	89	0.00715 30311+	39	+0.00816 17919+	89	+0.00357 65155+
40	0.00000 00000	90	0.00000 00000	40	−0.00795 77471+	90	−0.00353 67765
41	0.01552 73115	91	0.00699 58216+	41	+0.00776 36557+	91	+0.00349 79108
42	0.00000 00000	92	0.00000 00000	42	−0.00757 88068	92	−0.00345 98900+
43	0.01480 51109+	93	0.00684 53738+	43	+0.00740 25554+	93	+0.00342 26869
44	0.00000 00000	94	0.00000 00000	44	−0.00723 43155+	94	−0.00338 62753+
45	0.01414 71060+	95	0.00670 12607+	45	+0.00707 35530	95	+0.00335 06303+
46	0.00000 00000	96	0.00000 00000	46	−0.00691 97801	96	−0.00331 57279+
47	0.01354 51015	97	0.00656 30904	47	+0.00677 25507+	97	+0.00328 15452
48	0.00000 00000	98	0.00000 00000	48	−0.00663 14559+	98	−0.00324 80600+
49	0.01299 22402+	99	0.00643 05027+	49	+0.00649 61201	99	+0.00321 52513+
50	0.00000 00000	100	0.00000 00000	50	−0.00636 61977	100	−0.00318 30988+

A plus sign after an entry indicates that the eleventh decimal place is 5 or larger.

n	$\int_0^1 x^3 \sin n\pi x\, dx$	n	$\int_0^1 x^3 \sin n\pi x\, dx$	n	$\int_0^1 x^4 \sin n\pi x\, dx$	n	$\int_0^1 x^4 \sin n\pi x\, dx$
1	+0.18930 37484+	51	+0.00624 03977+	1	+0.12480 06795+	51	+0.00623 99115
2	−0.15915 49430+	52	−0.00612 13439+	2	−0.13496 62922+	52	−0.00611 99677
3	+0.10132 52902+	53	+0.00600 49803+	3	+0.09893 62877	53	+0.00600 45471
4	−0.07957 74715	54	−0.00589 46275	4	−0.07655 38901+	54	−0.00589 33986
5	+0.06262 99281	55	+0.00578 66770+	5	+0.06211 39035+	55	+0.00578 62893+
6	−0.05305 16476+	56	−0.00568 41051	6	−0.05215 57717	56	−0.00568 30032
7	+0.04509 67296+	57	+0.00558 36873+	7	+0.04490 86741	57	+0.00558 33390+
8	−0.03978 87357+	58	−0.00548 81014+	8	−0.03941 07881	58	−0.00548 71097
9	+0.03519 08019+	59	+0.00539 44546+	9	+0.03510 23204	59	+0.00539 41406
10	−0.03183 09886	60	−0.00530 51647+	10	−0.03163 74794	60	−0.00530 42688+
11	+0.02884 03381+	61	+0.00521 76264+	11	+0.02879 18760	61	+0.00521 73423
12	−0.02652 58238	62	−0.00513 40304	12	−0.02641 38393+	62	−0.00513 32184+
13	+0.02442 06566	63	+0.00505 20219	13	+0.02439 72970	63	+0.00505 17639+
14	−0.02273 64204	64	−0.00497 35919+	14	−0.02266 58995+	64	−0.00497 28537+
15	+0.02118 24350	65	+0.00489 66054	15	+0.02116 33230	65	+0.00489 63705
16	−0.01989 43678+	66	−0.00482 28770+	16	−0.01984 71244	66	−0.00482 22039+
17	+0.01869 78528	67	+0.00475 04648+	17	+0.01868 47237+	67	+0.00475 02504
18	−0.01768 38825+	68	−0.00468 10277	18	−0.01765 07019+	68	−0.00468 04123
19	+0.01673 43436	69	+0.00461 27940+	19	+0.01672 49394+	69	+0.00461 25977
20	−0.01591 54943	70	−0.00454 72840+	20	−0.01589 13056+	70	−0.00454 67199
21	+0.01514 36835+	71	+0.00448 28773+	21	+0.01513 67185+	71	+0.00448 26971+
22	−0.01446 86311+	72	−0.00442 09706	22	−0.01445 04578+	72	−0.00442 04521+
23	+0.01382 89573	73	+0.00436 00777+	23	+0.01382 36558	73	+0.00435 99119+
24	−0.01326 29119	74	−0.00430 14849	24	−0.01324 89138+	74	−0.00430 10074
25	+0.01272 41390+	75	+0.00424 38260	25	+0.01272 00108+	75	+0.00424 36731
26	−0.01224 26879	76	−0.00418 82879+	26	−0.01223 16780+	76	−0.00418 78471+
27	+0.01178 27008+	77	+0.00413 36120	27	+0.01177 94237+	77	+0.00413 34707+
28	−0.01136 82102	78	−0.00408 08959+	28	−0.01135 93951	78	−0.00408 04882
29	+0.01097 09134+	79	+0.00402 89774	29	+0.01096 82686+	79	+0.00402 88465+
30	−0.01061 03295	80	−0.00397 88735+	30	−0.01060 31625	80	−0.00397 84956
31	+0.01026 37304+	81	+0.00392 95089	31	+0.01026 15652+	81	+0.00392 93875+
32	−0.00994 71839	82	−0.00388 18278+	32	−0.00994 12785	82	−0.00388 14769
33	+0.00964 21643	83	+0.00383 48332+	33	+0.00964 03694	83	+0.00383 47204
34	−0.00936 20554+	84	−0.00378 94034	34	−0.00935 71320+	84	−0.00378 90769
35	+0.00909 15592+	85	+0.00374 46121	35	+0.00909 00548	85	+0.00374 45070+
36	−0.00884 19412+	86	−0.00370 12777	36	−0.00883 77937	86	−0.00370 09735
37	+0.00860 04230	87	+0.00365 85384	37	+0.00859 91496	87	+0.00365 84404+
38	−0.00837 65759+	88	−0.00361 71577+	38	−0.00837 30493+	88	−0.00361 68738
39	+0.00815 96171+	89	+0.00357 63325+	39	+0.00815 85297+	89	+0.00357 62410+
40	−0.00795 77471+	90	−0.00353 67765	40	−0.00795 47235+	90	−0.00353 65110+
41	+0.00776 17839+	91	+0.00349 77396	41	+0.00776 08480+	91	+0.00349 76540
42	−0.00757 88068	92	−0.00345 98900+	42	−0.00757 61949	92	−0.00345 96415+
43	+0.00740 09329	93	+0.00342 25265+	43	+0.00740 01216	93	+0.00342 24463+
44	−0.00723 43155+	94	−0.00338 62753+	44	−0.00723 20439	94	−0.00338 60424
45	+0.00707 21373	95	+0.00335 04799	45	+0.00707 14294+	95	+0.00335 04046+
46	−0.00691 97801	96	−0.00331 57279+	46	−0.00691 77920+	96	−0.00331 55092+
47	+0.00677 13082	97	+0.00328 14038+	47	+0.00677 06869	97	+0.00328 13331+
48	−0.00663 14559+	98	−0.00324 80600+	48	−0.00662 97062	98	−0.00324 78544+
49	+0.00649 50235+	99	+0.00321 51184	49	+0.00649 44753	99	+0.00321 50519
50	−0.00636 61977	100	−0.00318 30988+	50	−0.00636 46496+	100	−0.00318 29053+

A plus sign after an entry indicates that the eleventh decimal place is 5 or larger.

n	$\int_0^1 x^4 \sin n\pi x\, dx$	n	$\int_0^1 x^4 \sin n\pi x\, dx$	n	$\int_0^1 x^5 \sin n\pi x\, dx$	n	$\int_0^1 x^5 \sin n\pi x\, dx$
1	+0.08814 41278+	51	+0.00623 84532	1	+0.06541 08346+	51	+0.00623 65088
2	−0.11077 76414	52	−0.00611 85915	2	−0.09078 02206+	52	−0.00611 67575+
3	+0.09241 47642+	53	+0.00600 32476+	3	+0.08382 69803	53	+0.00600 15152
4	−0.07353 03088	54	−0.00589 21697	4	−0.06988 18080+	54	−0.00589 05320
5	+0.06061 60227+	55	+0.00578 51266	5	+0.05862 72138+	55	+0.00578 35762+
6	−0.05125 98957+	56	−0.00568 19013	6	−0.05011 58229+	56	−0.00568 04328+
7	+0.04435 38398+	57	+0.00558 22944	7	+0.04361 56163+	57	+0.00558 09016
8	−0.03903 28404	58	−0.00548 61179	8	−0.03854 08771	58	−0.00548 47961
9	+0.03483 95319+	59	+0.00539 31986	9	+0.03448 95901+	59	+0.00539 19426+
10	−0.03144 39702	60	−0.00530 33730	10	−0.03118 98792	60	−0.00530 21790
11	+0.02864 74636	61	+0.00521 64899+	11	+0.02845 50761	61	+0.00521 53535
12	−0.02630 18548+	62	−0.00513 24065	12	−0.02615 41180+	62	−0.00513 13243+
13	+0.02430 96406	63	+0.00505 09902+	13	+0.02419 28358+	63	+0.00504 99586
14	−0.02259 53787	64	−0.00497 21156	14	−0.02250 20800+	64	−0.00497 11317
15	+0.02110 61935	65	+0.00489 56660	15	+0.02103 00552+	65	+0.00489 47267
16	−0.01979 98809+	66	−0.00482 15308+	16	−0.01973 72636+	66	−0.00482 06337+
17	+0.01864 54470+	67	+0.00474 96071+	17	+0.01859 30965	67	+0.00474 87494+
18	−0.01761 75213+	68	−0.00467 97968+	18	−0.01757 34881	68	−0.00467 89765+
19	+0.01669 67903	69	+0.00461 20087+	19	+0.01665 92687+	69	+0.00461 12234+
20	−0.01586 71170	70	−0.00454 61557+	20	−0.01583 49880	70	−0.00454 54037+
21	+0.01511 58619	71	+0.00448 21565+	21	+0.01508 80594	71	+0.00448 14358
22	−0.01443 22846	72	−0.00441 99337	22	−0.01440 81296	72	−0.00441 92426+
23	+0.01380 77757+	73	+0.00435 94146	23	+0.01378 66064	73	+0.00435 87514+
24	−0.01323 49158	74	−0.00430 05298+	24	−0.01321 63009+	74	−0.00429 98933
25	+0.01270 76423	75	+0.00424 32145	25	+0.01269 11536	75	+0.00424 26030
26	−0.01222 06682	76	−0.00418 74063	26	−0.01220 60214	76	−0.00418 68187
27	+0.01176 96034	77	+0.00413 30469+	27	+0.01175 65114	77	+0.00413 24818+
28	−0.01135 05800	78	−0.00408 00804	28	−0.01133 88493	78	−0.00407 95368+
29	+0.01096 03420+	79	+0.00402 84541+	29	+0.01094 97744+	79	+0.00402 79309
30	−0.01059 59955	80	−0.00397 81176+	30	−0.01058 64556+	80	−0.00397 76138+
31	+0.01025 50752	81	+0.00392 90234+	31	+0.01024 64226+	81	+0.00392 85380+
32	−0.00993 53730+	82	−0.00388 11259+	32	−0.00992 75108+	82	−0.00388 06581
33	+0.00963 49887+	83	+0.00383 43820+	33	+0.00962 78152	83	+0.00383 39308+
34	−0.00935 22086+	84	−0.00378 87504	34	−0.00934 56527+	84	−0.00378 83152
35	+0.00908 55444+	85	+0.00374 41920	35	+0.00907 95311+	85	+0.00374 37719+
36	−0.00883 36461	86	−0.00370 06692+	36	−0.00882 81225	86	−0.00370 02637
37	+0.00859 53315+	87	+0.00365 81466	37	+0.00859 02412	87	+0.00365 77548+
38	−0.00836 95228	88	−0.00361 65898+	38	−0.00836 48257	88	−0.00361 62113
39	+0.00815 52693	89	+0.00357 59666	39	+0.00815 09223+	89	+0.00357 56006+
40	−0.00795 16999+	90	−0.00353 62456	40	−0.00794 76723+	90	−0.00353 58917+
41	+0.00775 80417	91	+0.00349 73972+	41	+0.00775 43001+	91	+0.00349 70549
42	−0.00757 35830	92	−0.00345 93930+	42	−0.00757 01035	92	−0.00345 90617+
43	+0.00739 76888	93	+0.00342 22058	43	+0.00739 44452+	93	+0.00342 18850+
44	−0.00722 97722+	94	−0.00338 58094	44	−0.00722 67457+	94	−0.00338 54988
45	+0.00706 93067+	95	+0.00335 01790	45	+0.00706 64766	95	+0.00334 98780+
46	−0.00691 58040	96	−0.00331 52905	46	−0.00691 31551+	96	−0.00331 49989+
47	+0.00676 88237+	97	+0.00328 11211+	47	+0.00676 63396+	97	+0.00328 08385
48	−0.00662 79564	98	−0.00324 76488+	48	−0.00662 56249+	98	−0.00324 73747+
49	+0.00649 28310+	99	+0.00321 48525	49	+0.00649 06388	99	+0.00321 45806
50	−0.00636 31015+	100	−0.00318 27118	50	−0.00636 10387	100	−0.00318 24538+

A plus sign after an entry indicates that the eleventh decimal place is 5 or larger.

n	$\int_0^1 x^6 \sin n\pi x\, dx$	n	$\int_0^1 x^6 \sin n\pi x\, dx$	n	$\int_0^1 x^7 \sin n\pi x\, dx$	n	$\int_0^1 x^7 \sin n\pi x\, dx$
1	+0.05038 38652	51	+0.00623 40798	1	+0.03995 47520+	51	+0.00623 11667+
2	−0.07497 40299+	52	−0.00611 44658+	2	−0.06257 63692+	52	−0.00611 17175+
3	+0.07489 13843+	53	+0.00599 93507+	3	+0.06646 72006+	53	+0.00599 67549
4	−0.06560 83879	54	−0.00588 84855	4	−0.06099 11393	54	−0.00588 60311
5	+0.05629 19526	55	+0.00578 16393+	5	+0.05368 24770	55	+0.00577 93162+
6	−0.04872 35532+	56	−0.00567 85978	6	−0.04712 75541	56	−0.00567 63968+
7	+0.04272 14223+	57	+0.00557 91613+	7	+0.04168 49672+	57	+0.00557 70741+
8	−0.03793 48981+	58	−0.00548 31443+	8	−0.03722 60747	58	−0.00548 11631+
9	+0.03406 03641+	59	+0.00539 03734	9	+0.03355 57886	59	+0.00538 84912
10	−0.03087 52065	60	−0.00530 06869	10	−0.03050 37065	60	−0.00529 88971+
11	+0.02821 76107	61	+0.00521 39335+	11	+0.02793 65162	61	+0.00521 22303+
12	−0.02597 06290	62	−0.00512 99719+	12	−0.02575 29170	62	−0.00512 83498
13	+0.02404 81424	63	+0.00504 86696	13	+0.02387 61902+	63	+0.00504 71234
14	−0.02238 60035	64	−0.00496 99021+	14	−0.02224 78624	64	−0.00496 84272+
15	+0.02093 55251	65	+0.00489 35530	15	+0.02082 29115+	65	+0.00489 21451
16	−0.01965 92724+	66	−0.00481 95125+	16	−0.01956 62752	66	−0.00481 81676+
17	+0.01852 80024	67	+0.00474 76777	17	+0.01845 03298+	67	+0.00474 63920+
18	−0.01751 86021+	68	−0.00467 79514	18	−0.01745 30683+	68	−0.00467 67216+
19	+0.01661 25642	69	+0.00461 02422	19	+0.01655 67714	69	+0.00460 90651
20	−0.01579 49186+	70	−0.00454 44639+	20	−0.01574 70302	70	−0.00454 33365+
21	+0.01505 34260+	71	+0.00448 05351+	21	+0.01501 20193	71	+0.00447 94547
22	−0.01437 79929+	72	−0.00441 83790	22	−0.01434 19501	72	−0.00441 73429
23	+0.01376 02207	73	+0.00435 79228	23	+0.01372 86552	73	+0.00435 69287
24	−0.01319 30693+	74	−0.00429 90977+	24	−0.01316 52698+	74	−0.00429 81434
25	+0.01267 05928+	75	+0.00424 18388+	25	+0.01264 59841	75	+0.00424 09221+
26	−0.01218 77376+	76	−0.00418 60843	26	−0.01216 58497+	76	−0.00418 52033
27	+0.01174 01805+	77	+0.00413 17757	27	+0.01172 06271+	77	+0.00413 09285+
28	−0.01132 42030	78	−0.00407 88575	28	−0.01130 66638	78	−0.00407 80425
29	+0.01093 65888+	79	+0.00402 72770	29	+0.01092 07967	79	+0.00402 64925+
30	−0.01057 45429	80	−0.00397 69841+	30	−0.01056 02733+	80	−0.00397 62287+
31	+0.01023 56241	81	+0.00392 79314	31	+0.01022 26877	81	+0.00392 72036
32	−0.00991 76918	82	−0.00388 00733+	32	−0.00990 59276+	82	−0.00387 93718+
33	+0.00961 88607+	83	+0.00383 33670	33	+0.00960 81314+	83	+0.00383 26905+
34	−0.00933 74643+	84	−0.00378 77712+	34	−0.00932 76520+	84	−0.00378 71186+
35	+0.00907 20238+	85	+0.00374 32469+	35	+0.00906 30270+	85	+0.00374 26171
36	−0.00882 12228+	86	−0.00369 97568	36	−0.00881 29536	86	−0.00369 91486+
37	+0.00858 38854	87	+0.00365 72652	37	+0.00857 62674	87	+0.00365 66778
38	−0.00835 89580	88	−0.00361 57382	38	−0.00835 19246+	88	−0.00361 51706
39	+0.00814 54940+	89	+0.00357 51433	39	+0.00813 89871	89	+0.00357 45946
40	−0.00794 26407	90	−0.00353 54494+	40	−0.00793 66088+	90	−0.00353 49188+
41	+0.00774 96274	91	+0.00349 66270+	41	+0.00774 40255+	91	+0.00349 61137
42	−0.00756 57563+	92	−0.00345 86477	42	−0.00756 05446	92	−0.00345 81509
43	+0.00739 03941+	93	+0.00342 14842	43	+0.00738 55371	93	+0.00342 10033
44	−0.00722 29644	94	−0.00338 51106	44	−0.00721 84306	94	−0.00338 46449
45	+0.00706 29416	95	+0.00334 95020	45	+0.00705 87029+	95	+0.00334 90508
46	−0.00690 98455+	96	−0.00331 46345	46	−0.00690 58770+	96	−0.00331 41972+
47	+0.00676 32367	97	+0.00328 04852	47	+0.00675 95158+	97	+0.00328 00613+
48	−0.00662 27117+	98	−0.00324 70321+	48	−0.00661 92184	98	−0.00324 66211+
49	+0.00648 79002+	99	+0.00321 42543	49	+0.00648 46162	99	+0.00321 38556
50	−0.00635 84611	100	−0.00318 21314	50	−0.00635 53699+	100	−0.00318 17445+

A plus sign after an entry indicates that the eleventh decimal place is 5 or larger.

n	$\int_0^1 x^2 \sin n\pi x\, dx$	n	$\int_0^1 x^3 \sin n\pi x\, dx$	n	$\int_0^1 x^4 \sin n\pi x\, dx$	n	$\int_0^1 x^5 \sin n\pi x\, dx$
1	+0.03243 25250	51	+0.00622 77709	1	+0.02683 49666+	51	+0.00622 38935
2	−0.05280 45386+	52	−0.00610 85135+	2	−0.04502 93306	52	−0.00610 48551+
3	+0.05888 85523	53	+0.00599 37286	3	+0.05222 70118	53	+0.00599 02730
4	−0.05631 11532	54	−0.00588 31696	4	−0.05176 88466	54	−0.00587 99020+
5	+0.05088 59865+	55	+0.00577 66078+	5	+0.04799 71615+	55	+0.00577 35150
6	−0.04587 22928+	56	−0.00567 38307+	6	−0.04350 16085+	56	−0.00567 09003+
7	+0.04052 58865	57	+0.00557 46406	7	+0.03926 67785	57	+0.00557 18615
8	−0.03642 55780+	58	−0.00547 88531+	8	−0.03554 54720+	58	−0.00547 62151
9	+0.03298 18634	59	+0.00538 62965+	9	+0.03234 56209	59	+0.00538 37901+
10	−0.03007 91336	60	−0.00529 68103	10	−0.02960 57050	60	−0.00529 44269+
11	+0.02761 40695	61	+0.00521 02443+	11	+0.02725 29616+	61	+0.00520 79761
12	−0.02550 25113+	62	−0.00512 64582+	12	−0.02522 11658	62	−0.00512 42978+
13	+0.02367 79863	63	+0.00504 53204	13	+0.02345 47261+	63	+0.00504 32611
14	−0.02208 83700	64	−0.00496 67074	14	−0.02190 83544	64	−0.00496 47430
15	+0.02069 27129	65	+0.00489 05033	15	+0.02054 55224	65	+0.00488 86281
16	−0.01945 86395+	66	−0.00481 65993	16	−0.01933 67959	66	−0.00481 48079
17	+0.01836 03475	67	+0.00474 48928+	17	+0.01825 83757+	67	+0.00474 31804
18	−0.01737 70914+	68	−0.00467 52875	18	−0.01729 09124	68	−0.00467 36494
19	+0.01649 20453+	69	+0.00460 76924	19	+0.01641 85709+	69	+0.00460 61244
20	−0.01569 14439	70	−0.00454 20217+	20	−0.01562 83029	70	−0.00454 05199+
21	+0.01496 39335	71	+0.00447 81946+	21	+0.01490 92812+	71	+0.00447 67553+
22	−0.01430 00763+	72	−0.00441 61346	22	−0.01425 24611	72	−0.00441 47543+
23	+0.01369 19699	73	+0.00435 57693+	23	+0.01365 02365	73	+0.00435 44449+
24	−0.01313 29513	74	−0.00429 70904	24	−0.01309 61718	74	−0.00429 57589+
25	+0.01261 73670	75	+0.00423 98530	25	+0.01258 47889+	75	+0.00423 86317
26	−0.01214 03905	76	−0.00418 41758	26	−0.01211 13988+	76	−0.00418 30020
27	+0.01169 78783	77	+0.00412 99405+	27	+0.01167 19663+	77	+0.00412 88118+
28	−0.01128 62543+	78	−0.00407 70920	28	−0.01126 30016	78	−0.00407 60061
29	+0.01090 24169	79	+0.00402 55776+	29	+0.01088 14721+	79	+0.00402 45324+
30	−0.01054 36630+	80	−0.00397 53477	30	−0.01052 47311+	80	−0.00397 43412
31	+0.01020 76271	81	+0.00392 63547+	31	+0.01019 04585+	81	+0.00392 53850+
32	−0.00989 22299+	82	−0.00387 85537	32	−0.00987 66126+	82	−0.00387 76190
33	+0.00959 56372	83	+0.00383 19015+	33	+0.00958 13900	83	+0.00383 10002
34	−0.00931 62244+	84	−0.00378 63575	34	−0.00930 31918	84	−0.00378 54879
35	+0.00905 25481	85	+0.00374 18825	35	+0.00904 05959+	85	+0.00374 10432
36	−0.00880 33213	86	−0.00369 84394	36	−0.00879 23335+	86	−0.00369 76290+
37	+0.00856 73929	87	+0.00365 59927	37	+0.00855 72686+	87	+0.00365 52099+
38	−0.00834 37306+	88	−0.00361 45085+	38	−0.00833 43818	88	−0.00361 37521+
39	+0.00813 14057	89	+0.00357 39546	39	+0.00812 27551+	89	+0.00357 32234
40	−0.00792 95806	90	−0.00353 42999+	40	−0.00792 15605+	90	−0.00353 35928
41	+0.00773 74979	91	+0.00349 55150	41	+0.00773 00486	91	+0.00349 48309
42	−0.00755 44712	92	−0.00345 75715	42	−0.00754 75397	92	−0.00345 69094+
43	+0.00737 98767	93	+0.00342 04423+	43	+0.00737 34162+	93	+0.00341 98014+
44	−0.00721 31466+	94	−0.00338 41016+	44	−0.00720 71155	94	−0.00338 34809+
45	+0.00705 37628+	95	+0.00334 85245+	45	+0.00704 81238	95	+0.00334 79232+
46	−0.00690 12516	96	−0.00331 36872+	46	−0.00689 59714	96	−0.00331 31045+
47	+0.00675 51788+	97	+0.00327 95669+	47	+0.00675 02277+	97	+0.00327 90020+
48	−0.00661 51464	98	−0.00324 61417	48	−0.00661 04976	98	−0.00324 55939+
49	+0.00648 07880+	99	+0.00321 33905+	49	+0.00647 64174+	99	+0.00321 28592
50	−0.00635 17665+	100	−0.00318 12933	50	−0.00634 76524	100	−0.00318 07777

A plus sign after an entry indicates that the eleventh decimal place is 5 or larger.

n	$\int_0^1 x^{10} \sin n\pi x\, dx$	n	$\int_0^1 x^{10} \sin n\pi x\, dx$	n	$\int_0^1 x \cos n\pi x\, dx$	n	$\int_0^1 x \cos n\pi x\, dx$
1	+0.02256 07137+	51	+0.00621 95362	1	−0.20264 23672+	51	−0.00007 79094
2	−0.03877 50299	52	−0.00610 07437+	2	+0.00000 00000	52	+0.00000 00000
3	+0.04643 67171+	53	+0.00598 63893+	3	−0.02251 58185+	53	−0.00007 21403+
4	−0.04748 39626	54	−0.00587 62296+	4	+0.00000 00000	54	+0.00000 00000
5	+0.04510 09950	55	+0.00577 00388	5	−0.00810 56946+	55	−0.00006 69892
6	−0.04155 87116	56	−0.00566 76066+	6	+0.00000 00000	56	+0.00000 00000
7	+0.03793 09680	57	+0.00556 87377	7	−0.00413 55585	57	−0.00006 23706+
8	−0.03459 87132	58	−0.00547 32497+	8	+0.00000 00000	58	+0.00000 00000
9	+0.03165 46968+	59	+0.00538 09727+	9	−0.00250 17576	59	−0.00005 82138
10	−0.02908 81005	60	−0.00529 17477+	10	+0.00000 00000	60	+0.00000 00000
11	+0.02685 61870	61	+0.00520 54263	11	−0.00167 47303	61	−0.00005 44591
12	−0.02491 08584	62	−0.00512 18692	12	+0.00000 00000	62	+0.00000 00000
13	+0.02320 77584	63	+0.00504 09460+	13	−0.00119 90672+	63	−0.00005 10562+
14	−0.02170 87582+	64	−0.00496 25346	14	+0.00000 00000	64	+0.00000 00000
15	+0.02038 20150	65	+0.00488 65198+	15	−0.00090 06327	65	−0.00004 79626+
16	−0.01920 12369+	66	−0.00481 27939	16	+0.00000 00000	66	+0.00000 00000
17	+0.01814 47812	67	+0.00474 12550+	17	−0.00070 11846+	67	−0.00004 51419+
18	−0.01719 48082+	68	−0.00467 18076+	18	+0.00000 00000	68	+0.00000 00000
19	+0.01633 65607	69	+0.00460 43614+	19	−0.00056 13361+	69	−0.00004 25629+
20	−0.01555 77722+	70	−0.00453 88313+	20	+0.00000 00000	70	+0.00000 00000
21	+0.01484 81925	71	+0.00447 51370	21	−0.00045 95065	71	−0.00004 01988
22	−0.01419 92075	72	−0.00441 32024	22	+0.00000 00000	72	+0.00000 00000
23	+0.01360 35379+	73	+0.00435 29558+	23	−0.00038 30668+	73	−0.00003 80263
24	−0.01305 49984+	74	−0.00429 43293	24	+0.00000 00000	74	+0.00000 00000
25	+0.01254 83049	75	+0.00423 72584	25	−0.00032 42277+	75	−0.00003 60253
26	−0.01207 89200+	76	−0.00418 16821+	26	+0.00000 00000	76	+0.00000 00000
27	+0.01164 29287+	77	+0.00412 75427	27	−0.00027 79730+	77	−0.00003 41781+
28	−0.01123 69368+	78	−0.00407 47850+	28	+0.00000 00000	78	+0.00000 00000
29	+0.01085 79887+	79	+0.00402 33571+	29	−0.00024 09540+	79	−0.00003 24695
30	−0.01050 34998+	80	−0.00397 32093+	30	+0.00000 00000	80	+0.00000 00000
31	+0.01017 12009	81	+0.00392 42945+	31	−0.00021 08661	81	−0.00003 08858+
32	−0.00985 90918	82	−0.00387 65678+	32	+0.00000 00000	82	+0.00000 00000
33	+0.00956 54036	83	+0.00382 99866	33	−0.00018 60811	83	−0.00002 94153+
34	−0.00928 85660	84	−0.00378 45100+	34	+0.00000 00000	84	+0.00000 00000
35	+0.00902 71809+	85	+0.00374 00994	35	−0.00016 54223	85	−0.00002 80473+
36	−0.00877 99994	86	−0.00369 67177+	36	+0.00000 00000	86	+0.00000 00000
37	+0.00854 59025	87	+0.00365 43296+	37	−0.00014 80221+	87	−0.00002 67726+
38	−0.00832 38850	88	−0.00361 29015+	38	+0.00000 00000	88	+0.00000 00000
39	+0.00811 30414	89	+0.00357 24011	39	−0.00013 32296+	89	−0.00002 55829
40	−0.00791 25539+	90	−0.00353 27976	40	+0.00000 00000	90	+0.00000 00000
41	+0.00772 16822	91	+0.00349 40616	41	−0.00012 05487	91	−0.00002 44707+
42	−0.00753 97543+	92	−0.00345 61649+	42	+0.00000 00000	92	+0.00000 00000
43	+0.00736 61593+	93	+0.00341 90806+	43	−0.00010 95956+	93	−0.00002 34295+
44	−0.00720 03403+	94	−0.00338 27829	44	+0.00000 00000	94	+0.00000 00000
45	+0.00704 17887+	95	+0.00334 72470	45	−0.00010 00703	95	−0.00002 24534
46	−0.00689 00391+	96	−0.00331 24492	46	+0.00000 00000	96	+0.00000 00000
47	+0.00674 46649	97	+0.00327 83667+	47	−0.00009 17348+	97	−0.00002 15370+
48	−0.00660 52741	98	−0.00324 49778+	48	+0.00000 00000	98	+0.00000 00000
49	+0.00647 15063	99	+0.00321 22616	49	−0.00008 43991+	99	−0.00002 06756+
50	−0.00634 30292+	100	−0.00318 01978+	50	+0.00000 00000	100	+0.00000 00000

A plus sign after an entry indicates that the eleventh decimal place is 5 or larger.

n	$\int_0^1 x^2 \cos n\pi x\, dx$	n	$\int_0^1 x^2 \cos n\pi x\, dx$	n	$\int_0^1 x^3 \cos n\pi x\, dx$	n	$\int_0^1 x^3 \cos n\pi x\, dx$
1	−0.20264 23672+	51	−0.00007 79094	1	−0.18077 17638+	51	−0.00011 68459
2	+0.05066 05918	52	+0.00007 49417	2	+0.07599 08877	52	+0.00011 24125+
3	−0.02251 58185+	53	−0.00007 21403+	3	−0.03225 28416	53	−0.00010 81949+
4	+0.01266 51479+	54	+0.00006 94932+	4	+0.01899 77219	54	+0.00010 42399
5	−0.00810 56946+	55	−0.00006 69892	5	−0.01196 14351+	55	−0.00010 04703+
6	+0.00562 89546	56	+0.00006 46181	6	+0.00844 34319+	56	+0.00009 69271+
7	−0.00413 55585	57	−0.00006 23706+	7	−0.00615 20292	57	−0.00009 35443+
8	+0.00316 62869+	58	+0.00006 02385	8	+0.00474 94304+	58	+0.00009 03577+
9	−0.00250 17576	59	−0.00005 82138	9	−0.00373 38600+	59	−0.00008 73105+
10	+0.00202 64236+	60	+0.00005 62895	10	+0.00303 96355	60	+0.00008 44343
11	−0.00167 47303	61	−0.00005 44591	11	−0.00250 36812+	61	−0.00008 16797+
12	+0.00140 72386+	62	+0.00005 27165	12	+0.00211 08579+	62	+0.00007 90748
13	−0.00119 90672+	63	−0.00005 10562+	13	−0.00179 42876	63	−0.00007 65765+
14	+0.00103 38896	64	+0.00004 94732	14	+0.00155 08344	64	+0.00007 42098+
15	−0.00090 06327	65	−0.00004 79626+	15	−0.00134 85156+	65	−0.00007 19371
16	+0.00079 15717	66	+0.00004 65202+	16	+0.00118 73576	66	+0.00006 97804
17	−0.00070 11846+	67	−0.00004 51419+	17	−0.00105 03020	67	−0.00006 77068+
18	+0.00062 54394	68	+0.00004 38240	18	+0.00093 81591	68	+0.00006 57360+
19	−0.00056 13361+	69	−0.00004 25629+	19	−0.00084 10590	69	−0.00006 38390
20	+0.00050 66059	70	+0.00004 13555+	20	+0.00075 99088+	70	+0.00006 20333+
21	−0.00045 95065	71	−0.00004 01988	21	−0.00068 56263	71	−0.00006 02934
22	+0.00041 86825+	72	+0.00003 90899+	22	+0.00062 80238+	72	+0.00005 86349
23	−0.00038 30668+	73	−0.00003 80263	23	−0.00057 41600+	73	−0.00005 70351+
24	+0.00035 18096+	74	+0.00003 70055	24	+0.00052 77144+	74	+0.00005 55083
25	−0.00032 42277+	75	−0.00003 60253	25	−0.00048 60263	75	−0.00005 40340+
26	+0.00029 97668	76	+0.00003 50835	26	+0.00044 96502	76	+0.00005 26252+
27	−0.00027 79730+	77	−0.00003 41781+	27	−0.00041 67277+	77	−0.00005 12637
28	+0.00025 84724	78	+0.00003 33074	28	+0.00038 77086	78	+0.00004 99611
29	−0.00024 09540+	79	−0.00003 24695	29	−0.00036 12569	79	−0.00004 87011
30	+0.00022 51581+	80	+0.00003 16628+	30	+0.00033 77372+	80	+0.00004 74943
31	−0.00021 08661	81	−0.00003 08858+	31	−0.00031 61658	81	−0.00004 63259+
32	+0.00019 78929	82	+0.00003 01371+	32	+0.00029 68394	82	+0.00004 52057+
33	−0.00018 60811	83	−0.00002 94153+	33	−0.00027 90178	83	−0.00004 41204
34	+0.00017 52961+	84	+0.00002 87191+	34	+0.00026 29442	84	+0.00004 30787
35	−0.00016 54223	85	−0.00002 80473+	35	−0.00024 80514	85	−0.00004 20687
36	+0.00015 63598+	86	+0.00002 73989	36	+0.00023 45397+	86	+0.00004 10983+
37	−0.00014 80221+	87	−0.00002 67726+	37	−0.00022 19675	87	−0.00004 01568+
38	+0.00014 03340	88	+0.00002 61676+	38	+0.00021 05010+	88	+0.00003 92514+
39	−0.00013 32296+	89	−0.00002 55829	39	−0.00019 97912+	89	−0.00003 83724
40	+0.00012 66514+	90	+0.00002 50175+	40	+0.00018 99772	90	+0.00003 75263+
41	−0.00012 05487	91	−0.00002 44707+	41	−0.00018 07794+	91	−0.00003 67043
42	+0.00011 48766	92	+0.00002 39416+	42	+0.00017 23149	92	+0.00003 59125
43	−0.00010 95956+	93	−0.00002 34295+	43	−0.00016 43574	93	−0.00003 51427
44	+0.00010 46706	94	+0.00002 29337	44	+0.00015 70059+	94	+0.00003 44005+
45	−0.00010 00703	95	−0.00002 24534	45	−0.00015 00754	95	−0.00003 36786+
46	+0.00009 57667	96	+0.00002 19881	46	+0.00014 36500+	96	+0.00003 29821+
47	−0.00009 17348+	97	−0.00002 15370+	47	−0.00013 75770+	97	−0.00003 23042
48	+0.00008 79524	98	+0.00002 10997+	48	+0.00013 19286	98	+0.00003 16496+
49	−0.00008 43991+	99	−0.00002 06756+	49	−0.00012 65773+	99	−0.00003 10122
50	+0.00008 10569	100	+0.00002 02642	50	+0.00012 15854	100	+0.00003 03963+

A plus sign after an entry indicates that the eleventh decimal place is 5 or larger.

n	$\int_0^1 x^4 \cos n\pi x\, dx$	n	$\int_0^1 x^4 \cos n\pi x\, dx$	n	$\int_0^1 x^5 \cos n\pi x\, dx$	n	$\int_0^1 x^5 \cos n\pi x\, dx$
1	−0.15890 11604+	51	−0.00015 57823+	1	−0.14028 57365	51	−0.00019 46824+
2	+0.08592 22102+	52	+0.00014 98497	2	+0.08815 40460+	52	+0.00018 72700
3	−0.04198 98646+	53	−0.00014 42495+	3	−0.04902 75551+	53	−0.00018 02729
4	+0.02436 78600+	54	+0.00013 89575+	4	+0.02925 67802+	54	+0.00017 36607
5	−0.01581 71756+	55	−0.00013 39514+	5	−0.01929 46793	55	−0.00016 74057
6	+0.01106 77985	56	+0.00012 92111+	6	+0.01359 71096+	56	+0.00016 14826
7	−0.00816 84999+	57	−0.00012 47180	7	−0.01008 44755	57	−0.00015 58683+
8	+0.00627 24217	58	+0.00012 04552+	8	+0.00776 53368+	58	+0.00015 05418+
9	−0.00496 59624+	59	−0.00011 64073	9	−0.00616 09819	59	−0.00014 54837+
10	+0.00402 82089+	60	+0.00011 25600+	10	+0.00500 44632+	60	+0.00014 06763
11	−0.00333 26322+	61	−0.00010 89004	11	−0.00414 48958+	61	−0.00013 61033
12	+0.00280 25953+	62	+0.00010 54163+	12	+0.00348 83918	62	+0.00013 17496+
13	−0.00238 95079+	63	−0.00010 20969	13	−0.00297 61534	63	−0.00012 76015+
14	+0.00206 13656+	64	+0.00009 89317+	14	+0.00256 86901+	64	+0.00012 36463+
15	−0.00179 63986+	65	−0.00009 59115+	15	−0.00223 94366+	65	−0.00011 98722
16	+0.00157 93839+	66	+0.00009 30275+	16	+0.00196 95305+	66	+0.00011 62682+
17	−0.00139 94193+	67	−0.00009 02717	17	−0.00174 55970+	67	−0.00011 28243+
18	+0.00124 85317+	68	+0.00008 76365+	18	+0.00155 77308+	68	+0.00010 95312+
19	−0.00112 07818	69	−0.00008 51150+	19	−0.00139 86193	69	−0.00010 63802+
20	+0.00101 16719	70	+0.00008 27009	20	+0.00126 26650+	70	+0.00010 33633
21	−0.00091 77461	71	−0.00008 03879+	21	−0.00114 56019+	71	−0.00010 04728+
22	+0.00083 63133+	72	+0.00007 81707+	22	+0.00104 40770	72	+0.00009 77019+
23	−0.00076 52532+	73	−0.00007 60440	23	−0.00095 54677	73	−0.00009 50441+
24	+0.00070 28767	74	+0.00007 40028+	24	+0.00087 76676	74	+0.00009 24933
25	−0.00064 78248	75	−0.00007 20428	25	−0.00080 89036	75	−0.00009 00438
26	+0.00059 89944+	76	+0.00007 01596	26	+0.00074 80691	76	+0.00008 76903
27	−0.00055 54825	77	−0.00006 83493	27	−0.00069 37742+	77	−0.00008 54279
28	+0.00051 65439+	78	+0.00006 66081+	28	+0.00064 51788+	78	+0.00008 32519
29	−0.00048 15597+	79	−0.00006 49327	29	−0.00060 15146+	79	−0.00008 11580
30	+0.00045 00121+	80	+0.00006 33197	30	+0.00056 21350	80	+0.00007 91421
31	−0.00042 14655	81	−0.00006 17660+	31	−0.00052 64986+	81	−0.00007 72004
32	+0.00039 55509	82	+0.00006 02689	32	+0.00049 41449	82	+0.00007 53293
33	−0.00037 19545	83	−0.00005 88255	33	−0.00046 46836+	83	−0.00007 35254
34	+0.00035 04079+	84	+0.00005 74333+	34	+0.00043 77794+	84	+0.00007 17855
35	−0.00033 06804+	85	−0.00005 60900+	35	−0.00041 31455	85	−0.00007 01066+
36	+0.00031 25730	86	+0.00005 47933	36	+0.00039 05329	86	+0.00006 84860
37	−0.00029 59129	87	−0.00005 35410	37	−0.00036 97268+	87	−0.00006 69209
38	+0.00028 05499	88	+0.00005 23312	38	+0.00035 05397	88	+0.00006 54088+
39	−0.00026 63528+	89	−0.00005 11619	39	−0.00033 28080+	89	−0.00006 39475
40	+0.00025 32067	90	+0.00005 00313+	40	+0.00031 63880+	90	+0.00006 25345+
41	−0.00024 10102	91	−0.00004 89379	41	−0.00030 11538	91	−0.00006 11679
42	+0.00022 96740+	92	+0.00004 78799	42	+0.00028 69936	92	+0.00005 98455+
43	−0.00021 91192	93	−0.00004 68558+	43	−0.00027 38090	93	−0.00005 85656+
44	+0.00020 92755+	94	+0.00004 58642+	44	+0.00026 15122+	94	+0.00005 73264
45	−0.00020 00805	95	−0.00004 49038+	45	−0.00025 00255+	95	−0.00005 61260+
46	+0.00019 14784	96	+0.00004 39733	46	+0.00023 92792	96	+0.00005 49630
47	−0.00018 34192+	97	−0.00004 30713+	47	−0.00022 92110	97	−0.00005 38357
48	+0.00017 58584	98	+0.00004 21969	48	+0.00021 97650	98	+0.00005 27427+
49	−0.00016 87555+	99	−0.00004 13488	49	−0.00021 08910+	99	−0.00005 16827+
50	+0.00016 20744+	100	+0.00004 05260	50	+0.00020 25438	100	+0.00005 06544

A plus sign after an entry indicates that the eleventh decimal place is 5 or larger.

n	$\int_0^1 x^6 \cos n\pi x\, dx$	n	$\int_0^1 x^6 \cos n\pi x\, dx$	n	$\int_0^1 x^7 \cos n\pi x\, dx$	n	$\int_0^1 x^7 \cos n\pi x\, dx$
1	−0.12492 54920+	51	−0.00023 35461+	1	−0.11226 37768+	51	−0.00027 23644
2	+0.08668 87251	52	+0.00022 46566+	2	+0.08352 74122+	52	+0.00026 20012
3	−0.05336 59131	53	−0.00021 62650+	3	−0.05562 35587+	53	−0.00025 22182+
4	+0.03336 61055+	54	+0.00020 83349+	4	+0.03654 66473+	54	+0.00024 29730
5	−0.02239 39461	55	−0.00020 08330	5	−0.02508 55990	55	−0.00023 42267
6	+0.01595 23619	56	+0.00019 37290+	6	+0.01809 40534+	56	+0.00022 59442
7	−0.01189 99559	57	−0.00018 69953+	7	−0.01359 86510+	57	−0.00021 80932+
8	+0.00920 09566+	58	+0.00018 06067	8	+0.01056 56714+	58	+0.00021 06443+
9	−0.00731 89183	59	−0.00017 45398+	9	−0.00843 24727	59	−0.00020 35705+
10	+0.00595 68281	60	+0.00016 87736	10	+0.00687 95184	60	+0.00019 68471
11	−0.00494 04720	61	−0.00016 32883+	11	−0.00571 57828	61	−0.00019 04512
12	+0.00416 25571+	62	+0.00015 80662+	12	+0.00482 22463	62	+0.00018 43620
13	−0.00355 42240+	63	−0.00015 30906	13	−0.00412 17946	63	−0.00017 85601+
14	+0.00306 97005	64	+0.00014 83462+	14	+0.00356 28431	64	+0.00017 30278+
15	−0.00267 76297+	65	−0.00014 38190+	15	−0.00310 98594+	65	−0.00016 77486+
16	+0.00235 59623	66	+0.00013 94959	16	+0.00273 77615+	66	+0.00016 27074
17	−0.00208 88352	67	−0.00013 53648	17	−0.00242 84426	67	−0.00015 78899+
18	+0.00186 46050	68	+0.00013 14145	18	+0.00216 85783	68	+0.00015 32833
19	−0.00167 45715+	69	−0.00012 76346	19	−0.00194 81896+	69	−0.00014 88753+
20	+0.00151 21299+	70	+0.00012 40154+	20	+0.00175 96875+	70	+0.00014 46547+
21	−0.00137 21938+	71	−0.00012 05480+	21	−0.00159 72181	71	−0.00014 06111
22	+0.00125 07954+	72	+0.00011 72240+	22	+0.00145 62091	72	+0.00013 67346+
23	−0.00114 48034	73	−0.00011 40356	23	−0.00133 30478	73	−0.00013 30162+
24	+0.00105 17198	74	+0.00011 09755+	24	+0.00122 48496	74	+0.00012 94475
25	−0.00096 95327	75	−0.00010 80369+	25	−0.00112 92889+	75	−0.00012 60204+
26	+0.00089 66070+	76	+0.00010 52136	26	+0.00104 44746+	76	+0.00012 27276+
27	−0.00083 16030+	77	−0.00010 24994+	27	−0.00096 88558+	77	−0.00011 95622+
28	+0.00077 34145	78	+0.00009 98889+	28	+0.00090 11514	78	+0.00011 65177+
29	−0.00072 11216+	79	−0.00009 73769+	29	−0.00084 02955	79	−0.00011 35880
30	+0.00067 39546+	80	+0.00009 49585	30	+0.00078 53956+	80	+0.00011 07674
31	−0.00063 12653	81	−0.00009 26290+	31	−0.00073 57000+	81	−0.00010 80505+
32	+0.00059 25046+	82	+0.00009 03842+	32	+0.00069 05717	82	+0.00010 54324
33	−0.00055 72052	83	−0.00008 82201	33	−0.00064 94681+	83	−0.00010 29083
34	+0.00052 49671	84	+0.00008 61327	34	+0.00061 19250	84	+0.00010 04737+
35	−0.00049 54464+	85	−0.00008 41185+	35	−0.00057 75429+	85	−0.00009 81245+
36	+0.00046 83464	86	+0.00008 21742	36	+0.00054 59771	86	+0.00009 58567+
37	−0.00044 34095	87	−0.00008 02965	37	−0.00051 69283+	87	−0.00009 36667
38	+0.00042 04115+	88	+0.00007 84824	38	+0.00049 01361	88	+0.00009 15508+
39	−0.00039 91567+	89	−0.00007 67291	39	−0.00046 53727+	89	−0.00008 95058+
40	+0.00037 94734	90	+0.00007 50339+	40	+0.00044 24386+	90	+0.00008 75286+
41	−0.00036 12103	91	−0.00007 33943	41	−0.00042 11580+	91	−0.00008 56162
42	+0.00034 42341	92	+0.00007 18078	42	+0.00040 13758	92	+0.00008 37657+
43	−0.00032 84267	93	−0.00007 02722	43	−0.00038 29546	93	−0.00008 19746+
44	+0.00031 36833+	94	+0.00006 87853+	44	+0.00036 57724	94	+0.00008 02404
45	−0.00029 99105+	95	−0.00006 73452	45	−0.00034 97206	95	−0.00007 85606
46	+0.00028 70250+	96	+0.00006 59498	46	+0.00033 47022+	96	+0.00007 69329+
47	−0.00027 49522+	97	−0.00006 45973	47	−0.00032 06305	97	−0.00007 53554
48	+0.00026 36252	98	+0.00006 32860	48	+0.00030 74275+	98	+0.00007 38258+
49	−0.00025 29838	99	−0.00006 20142	49	−0.00029 50232+	99	−0.00007 23424
50	+0.00024 29737+	100	+0.00006 07803+	50	+0.00028 33545	100	+0.00007 09032+

A plus sign after an entry indicates that the eleventh decimal place is 5 or larger.

n	$\int_0^1 x^5 \cos n\pi x\, dx$	n	$\int_0^1 x^5 \cos n\pi x\, dx$	n	$\int_0^1 x^9 \cos n\pi x\, dx$	n	$\int_0^1 x^9 \cos n\pi x\, dx$
1	−0.10174 39406	51	−0.00031 11281+	1	−0.09291 23427	51	−0.00034 98284
2	+0.07967 47079	52	+0.00029 92954	2	+0.07563 69301+	52	+0.00033 65308
3	−0.05641 91122	53	−0.00028 81247	3	−0.05623 44251	53	−0.00032 39767
4	+0.03882 81652	54	+0.00027 75676+	4	+0.04032 98927	54	+0.00031 21118
5	−0.02734 02610	55	−0.00026 75801	5	−0.02915 55226+	55	−0.00030 08865+
6	+0.02000 15551+	56	+0.00025 81218+	6	+0.02166 36740+	56	+0.00029 02558+
7	−0.01516 42710+	57	−0.00024 91561+	7	−0.01658 54446+	57	−0.00028 01784
8	+0.01184 94276	58	+0.00024 06494	8	+0.01304 39493	58	+0.00027 06165
9	−0.00949 43460	59	−0.00023 25708+	9	−0.01049 84531+	59	−0.00026 15356
10	+0.00776 77050+	60	+0.00022 48921+	10	+0.00861 70370	60	+0.00025 29040+
11	−0.00646 72504	61	−0.00021 75874+	11	−0.00719 16801+	61	−0.00024 46926
12	+0.00546 49387	62	+0.00021 06328	12	+0.00608 82761	62	+0.00023 68745
13	−0.00467 69399	63	−0.00020 40062+	13	−0.00521 78795+	63	−0.00022 94250+
14	+0.00404 66940	64	+0.00019 76874	14	+0.00451 98942	64	+0.00022 23213+
15	−0.00353 50072+	65	−0.00019 16576	15	−0.00395 20170+	65	−0.00021 55424+
16	+0.00311 40694	66	+0.00018 58994	16	+0.00348 40560	66	+0.00020 90687+
17	−0.00276 37281+	67	−0.00018 03968	17	−0.00309 40306+	67	−0.00020 28823
18	+0.00246 91040+	68	+0.00017 51349	18	+0.00276 56500	68	+0.00019 69663+
19	−0.00221 90248+	69	−0.00017 00998	19	−0.00248 66436+	69	−0.00019 13053
20	+0.00200 49741+	70	+0.00016 52787	20	+0.00224 76337+	70	+0.00018 58847+
21	−0.00182 03711	71	−0.00016 06596+	21	−0.00204 13577	71	−0.00018 06913
22	+0.00166 00670+	72	+0.00015 62315	22	+0.00186 21227+	72	+0.00017 57124
23	−0.00151 99884	73	−0.00015 19839	23	−0.00170 54175+	73	−0.00017 09364
24	+0.00139 68785	74	+0.00014 79071+	24	+0.00156 76305+	74	+0.00016 63525
25	−0.00128 81093+	75	−0.00014 39922+	25	−0.00144 58437+	75	−0.00016 19504+
26	+0.00119 15416	76	+0.00014 02306+	26	+0.00133 76791	76	+0.00015 77207+
27	−0.00110 54197	77	−0.00013 66145+	27	−0.00124 11834	77	−0.00015 36546+
28	+0.00102 82922+	78	+0.00013 31365	28	+0.00115 47406	78	+0.00014 97436+
29	−0.00095 89510+	79	−0.00012 97896	29	−0.00107 70042+	79	−0.00014 59801
30	+0.00089 63838	80	+0.00012 65672+	30	+0.00100 68456+	80	+0.00014 23566+
31	−0.00083 97374	81	−0.00012 34634+	31	−0.00094 33128	81	−0.00013 88663+
32	+0.00078 82886+	82	+0.00012 04724	32	+0.00088 55984+	82	+0.00013 55029
33	−0.00074 14213+	83	−0.00011 75887+	33	−0.00083 30144	83	−0.00013 22601
34	+0.00069 86079+	84	+0.00011 48073+	34	+0.00078 49710+	84	+0.00012 91323
35	−0.00065 93945	85	−0.00011 21234+	35	−0.00074 09611	85	−0.00012 61141+
36	+0.00062 33889	86	+0.00010 95326	36	+0.00070 05460+	86	+0.00012 32005+
37	−0.00059 02509+	87	−0.00010 70305	37	−0.00066 33452	87	−0.00012 03867+
38	+0.00055 96842+	88	+0.00010 46131	38	+0.00062 90270	88	+0.00011 76682
39	−0.00053 14297+	89	−0.00010 22767	39	−0.00059 73015+	89	−0.00011 50407
40	+0.00050 52602	90	+0.00010 00177	40	+0.00056 79143+	90	+0.00011 25002+
41	−0.00048 09755+	91	−0.00009 78327+	41	−0.00054 06414	91	−0.00011 00429+
42	+0.00045 83992+	92	+0.00009 57185+	42	+0.00051 52849	92	+0.00010 76653+
43	−0.00043 73747+	93	−0.00009 36721+	43	−0.00049 16695	93	−0.00010 53639
44	+0.00041 77632	94	+0.00009 16907	44	+0.00046 96396	94	+0.00010 31355
45	−0.00039 94408+	95	−0.00008 97714+	45	−0.00044 90564+	95	−0.00010 09770
46	+0.00038 22972	96	+0.00008 79118	46	+0.00042 97962+	96	+0.00009 88855+
47	−0.00036 62333	97	−0.00008 61093+	47	−0.00041 17481	97	−0.00009 68584
48	+0.00035 11604	98	+0.00008 43617+	48	+0.00039 48124+	98	+0.00009 48929+
49	−0.00033 69987+	99	−0.00008 26668	49	−0.00037 88998	99	−0.00009 29867
50	+0.00032 36763	100	+0.00008 10224+	50	+0.00036 39294	100	+0.00009 11373

A plus sign after an entry indicates that the eleventh decimal place is 5 or larger.

n	$\int_0^1 x^{10} \cos n\pi x\, dx$	n	$\int_0^1 x^{10} \cos n\pi x\, dx$	n	$\int_0^1 x^{10} \cos n\pi x\, dx$	n	$\int_0^1 x^{10} \cos n\pi x\, dx$
1	−0.08541 83519	26	+0.00148 27607+	51	−0.00038 84562	76	+0.00017 51961+
2	+0.07166 64055	27	−0.00137 60378+	52	+0.00037 36991+	77	−0.00017 06807
3	−0.05541 45806	28	+0.00128 04017	53	−0.00035 97666	78	+0.00016 63375+
4	+0.04119 63391+	29	−0.00119 43724+	54	+0.00034 65983	79	−0.00016 21580
5	−0.03055 59420+	30	+0.00111 67086+	55	−0.00033 41394	80	+0.00015 81340
6	+0.02307 83201	31	−0.00104 63624+	56	+0.00032 23399	81	−0.00015 42578+
7	−0.01785 57197	32	+0.00098 24448	57	−0.00031 11541	82	+0.00015 05224+
8	+0.01414 30939+	33	−0.00092 41973	58	+0.00030 05402	83	−0.00014 69211
9	−0.01143 99232	34	+0.00087 09699+	59	−0.00029 04599	84	+0.00014 34474
10	+0.00942 37885+	35	−0.00082 22031+	60	+0.00028 08780+	85	−0.00014 00954
11	−0.00788 62610	36	+0.00077 74129+	61	−0.00027 17623	86	+0.00013 68595
12	+0.00669 01220	37	−0.00073 61792	62	+0.00026 30830	87	−0.00013 37344
13	−0.00574 29778+	38	+0.00069 81358	63	−0.00025 48126+	88	+0.00013 07151
14	+0.00498 11755+	39	−0.00066 29623+	64	+0.00024 69260+	89	−0.00012 77968+
15	−0.00435 98952+	40	+0.00063 03777+	65	−0.00023 93997+	90	+0.00012 49752+
16	+0.00384 69333	41	−0.00060 01343+	66	+0.00023 22122+	91	−0.00012 22460+
17	−0.00341 87185	42	+0.00057 20134+	67	−0.00022 53434+	92	+0.00011 96052+
18	+0.00305 77046+	43	−0.00054 58212	68	+0.00021 87748	93	−0.00011 70490+
19	−0.00275 06281	44	+0.00052 13854+	69	−0.00021 24891	94	+0.00011 45739+
20	+0.00248 73216+	45	−0.00049 85527+	70	+0.00020 64703	95	−0.00011 21765
21	−0.00225 98912+	46	+0.00047 71860+	71	−0.00020 07035+	96	+0.00010 98535
22	+0.00206 21360	47	−0.00045 71626	72	+0.00019 51749+	97	−0.00010 76019
23	−0.00188 91327	48	+0.00043 83722	73	−0.00018 98716	98	+0.00010 54188
24	+0.00173 69337	49	−0.00042 07158+	74	+0.00018 47814	99	−0.00010 33015
25	−0.00160 23451	50	+0.00040 41041	75	−0.00017 98931	100	+0.00010 12473

A plus sign after an entry indicates that the eleventh decimal place is 5 or larger.

THE SOLUTION OF LINEAR OPERATOR EQUATIONS

By F. J. Murray

§1. We consider the equation $Tf = g$, where T is a linear operator, for instance, an integral operator

$$Tf = \int_0^{2\pi} K(x, y)f(x)\,dx = g(y). \tag{1}$$

We suppose that $g(y)$ is given and is such that $\int_0^{2\pi} |g|^2\,dy$ is finite. Our problem is to find an $f(x)$ which is also integrable squared and satisfies the equation. We permit K, f and g to be complex valued. In what follows, $\mathfrak{L}_2$ denotes the set of functions for which $\int_0^{2\pi} |h|^2\,dx$ is finite, (f, g) denotes the integral $\int_0^{2\pi} f(x)\bar{g}(x)\,dx$, and $||f||$, the "norm" of f, denotes $(f, f)^{\frac{1}{2}}$.

We present here a method for solving the equation $Tf = g$, which previously appeared in the author's thesis. (Cf. *F. J. Murray* (1).) This method, by considering no more than two numerical series, settles the existence question for a given g and presents a solution as a series convergent in the mean, when there is at least one solution. Frequently the discussion of the first of these numerical series may be short-circuited. The method is elementary in the sense that no knowledge of the characteristic values or characteristic functions of the operator is needed.

In §2, the difficulties involved in solving such an equation are pointed out and the method mentioned above described. §3 compares this method with the Schmidt method for solving an infinite set of equation in an infinite number of unknowns. §4 describes the structure of linear operators upon which the method is based and contains a heuristic justification for it. §5 discusses the limitations within which the method is applicable. The remainder of the paper is concerned with an actual numerical example chosen at random but illustrative of the theory of determinants which would be applied in general.

§2. One method of solving the equation $Tf = g$, involves the Gram-Schmidt process of "orthonormalizing" a sequence of functions $f_1(x), f_2(x), \cdots$. This is a process of deriving another set of functions $\varphi_1, \varphi_2, \cdots$, which has the orthonormality property, i.e. $(\varphi_i, \varphi_j) = 0$ if $i \neq j$ and $(\varphi_i, \varphi_i) = 1$. This process may be described as taking place in two steps. In the first step, one expels from the sequence any f_n which is linearly dependent on $f_1, \cdots, f_{n-1}$. The result is a sequence $g_1, g_2, \cdots$ in which each function is linearly independent of the preceding functions. The second step consists in forming the functions

$$\varphi_n = \begin{vmatrix} g_1(x) & , \cdots, & g_n(x) \\ (g_1, g_1) & , \cdots, & (g_1, g_n) \\ \vdots & & \\ (g_{n-1}, g_1), & \cdots, & (g_{n-1}, g_n) \end{vmatrix} \div |A_{n-1}A_n|^{\frac{1}{2}} \tag{2}$$

where A_n is the determinant

$$A_n = |(g_i, g_j)|_{i,j=1,\cdots,n}. \tag{3}$$

One might attempt to solve $Tf = g$, by means of this process as follows. Take a complete orthonormal set $\varphi_1, \varphi_2, \cdots$ (An orthonormal set is complete if every function f in $\mathfrak{L}_2$, can be expressed as a series $\sum_{\alpha=1}^{\infty} x_\alpha \varphi_\alpha$ where $x_n = (f, \varphi_n)$ and $\sum_{\alpha=1}^{\infty} |x_\alpha|^2 < \infty$.) Form $T\varphi_1, T\varphi_2, \cdots$. Orthonormalize the latter into a set $\psi_1, \psi_2, \cdots$. Neglecting the possibility that $T(\sum_{\alpha=1}^{n} a_\alpha \varphi_\alpha) = 0$ for some finite linear combination $\sum_{\alpha=1}^{n} a_\alpha \varphi_\alpha$, we would then have constant $c_{i,j}$, $i \geqq j$, such that

$$\begin{aligned}
\psi_1 &= c_{1,1} T\varphi_1 \\
\psi_2 &= c_{2,1} T\varphi_1 + c_{2,2} T\varphi_2 \\
&\vdots \qquad\qquad \vdots \\
\psi_n &= \textstyle\sum_{\alpha=1}^{n} c_{n,\alpha} T\varphi_\alpha = T(\sum_{\alpha=1}^{n} c_{n,\alpha}\varphi_\alpha).
\end{aligned}$$

Consider then the equation $Tf = g$. It can be shown that a necessary condition that there be an f such that $Tf = g$ is that

$$g = \textstyle\sum_{\alpha=1}^{\infty} y_\alpha \psi_\alpha \tag{4}$$

for $y_n = (g, \psi_n)$.[1] We have, of course, $\sum_{\alpha=1}^{\infty} |y_\alpha|^2 < \infty$. (4) is equivalent to

$$(g, g) = \textstyle\sum_{\alpha=1}^{\infty} |(g, \psi_\alpha)|^2. \tag{4'}$$

Proceeding formally, we have

$$\begin{aligned}
g &= \textstyle\sum_{\alpha=1}^{\infty} y_\alpha \psi_\alpha = \sum_{\alpha=1}^{\infty} y_\alpha T(\sum_{\beta=1}^{\alpha} c_{\alpha,\beta}\varphi_\beta) \\
&= T\textstyle\sum_{\beta=1}^{\infty}(\sum_{\alpha=\beta}^{\infty} y_\alpha c_{\alpha,\beta})\varphi_\beta) = T(\sum_{\beta=1}^{\infty} x_\beta \varphi_\beta)
\end{aligned}$$

where $x_\beta = \sum_{\alpha=\beta}^{\infty} y_\alpha c_{\alpha,\beta}$.

Thus x_β is determined by an infinite series which need not converge. What is worse, this convergence failure does not indicate that there is no f such that $Tf = g$. It just tells us that our present method fails to find it.

For a more useful method, we must introduce the adjoint operator T^*. This is the operator which has the property that

$$(Tf, g) = (f, T^*g) \tag{5}$$

for all f and g. In the case of our integral operator, it is

$$T^*f = \int_0^{2\pi} \bar{K}(y, x) f(x)\, dx. \tag{6}$$

Now suppose we choose the φ's, not as an arbitrary complete orthonormal set but in the following way. Let $\chi_1, \chi_2, \cdots$ be a complete orthonormal set. Form $T^*\chi_1, T^*\chi_2, \cdots$. Now orthonormalize these into $\varphi_1, \varphi_2, \cdots$. In the first step of this process, the sequence $T^*\chi_1, T^*\chi_2, \cdots$ has been converted into

[1] In the terminology introduced later, this is the condition that g be in $[R]$. This corresponds to the usual condition on a finite system of linear equations on the coefficient matrix and augmented matrix. Cf. M. Bocher, p. 46, Theorem I.

the linearly independent set, $T^*\chi_{1'}$, $T^*\chi_{2'}$, $\cdots$ and from the second step we see that

$$\varphi_n = \sum_{\alpha=1}^{n} b_{n,\alpha} T^*\chi_{\alpha'} = T^*w_n, \tag{7}$$

where

$$w_n = \sum_{\alpha=1}^{n} b_{n,\alpha}\chi_{\alpha'}. \tag{8}$$

Suppose then that $g = Tf$, for an $f = \sum_{\alpha=1}^{\infty} x_\alpha\varphi_\alpha$. It is a well-known property of orthonormal sets that $x_n = (f, \varphi_n)$. Thus $x_n = (f, \varphi_n) = (f, T^*\omega_n) = (Tf, \omega_n) = (g, \omega_n)$. In this case, then we have no difficulty concerning the existence of x_n and f exists if and only if $\sum_{\alpha=1}^{\infty} |x_\alpha|^2 < \infty$.

A theoretical discussion shows that this second method is completely valid. If there is an f such that $Tf = g$, then there is one such that $f = \sum_{\alpha=1}^{\infty} x_\alpha\varphi_\alpha$. For sufficiency, however the "range condition" (4) or (4′) must also be considered. Completely stated the method is as follows.

Let $\chi_1, \chi_2, \cdots$ be a complete orthonormal set (in the range space). Form $T^*\chi_1, T^*\chi_2, \cdots$ and orthonormalize the latter into $\varphi_1, \varphi_2, \cdots$. Let ω_n be such that $\varphi_n = T^*\omega_n$ as in (8) above. Form $T\varphi_1, T\varphi_2, \cdots$ and orthonormalize into $\psi_1, \psi_2, \cdots$. Then the following two conditions are necessary and together are sufficient that there be an f such that $Tf = g$:

(a) $$g = \sum_{\alpha=1}^{\infty} y_\alpha\psi_\alpha \quad \text{or} \quad (g, g) = \sum_{\alpha=1}^{\infty} |(g, \psi_\alpha)|^2$$

(b) $$\sum_{\alpha=1}^{\infty} |(g, w_\alpha)|^2 < \infty.$$

When these two conditions are satisfied, $f_0 = \sum_{\alpha=1}^{\infty} (g, \omega_\alpha)\varphi_\alpha$ is such that $Tf_0 = g$. Furthermore, this f_0 is the solution of $Tf = g$ for g fixed, which has the least norm, $\|f\|$.

Let f be any other solution of $Tf = g$. Then $Tf - Tf_0 = 0$ or $T(f - f_0) = 0$. Thus any other solution, f, is obtained by adding an h such that $Th = 0$ to f_0, i.e. $f = f_0 + h$.

Thus to complete the solution of our equation, we must determine $\mathfrak{N}$, the set of h's such that $Th = 0$. This can be done as follows:[2]

We first point out that $\mathfrak{N}$ is exactly the set of elements h such that $(h, T^*g) = 0$ for every g. (Note that $Th = 0$, is equivalent to $(h, T^*g) = (Th, g) = (0, g) = 0$, for every g.) It is enough that $(h, T^*\chi_n) = 0$ for a complete orthonormal set $\chi_1, \chi_2, \cdots$. Orthonormalizing yields the equivalent condition $(h, \varphi_n) = 0$ for $n = 1, 2, \cdots$. If the $\varphi_1, \varphi_2, \cdots$ form a complete orthonormal set then $\mathfrak{N}$ is just the function 0. (We write this $\mathfrak{N} = (0)$.) Otherwise we can complete the set $\varphi_1, \varphi_2, \cdots$, by adding functions $\nu_1, \nu_2, \cdots$. $\mathfrak{N}$ consists of functions $h = \sum_{\alpha=1}^{\infty} z_\alpha\nu_\alpha$.

(The completion process may be carried out by means of a known complete orthonormal set, $\tau_1, \tau_2, \cdots$. Form

$$h_n = \tau_n - \sum_{\alpha=1}^{\infty} (\tau_n, \varphi_\alpha)\varphi_\alpha.$$

[2] This method is given for the sake of completeness. It is not a desirable method from the calculational point of view and in most instances easier and more direct methods of obtaining $\mathfrak{N}$ are known.

One can readily show that $(h_n, \varphi_m) = 0$ for all n and m. Hence h_n is in $\mathfrak{N}$. We then orthonormalize the sequence $h_1, h_2, \cdots$ to $\nu_1, \nu_2, \cdots$. The combined set $\varphi_1, \varphi_2, \cdots; \nu_1, \nu_2, \cdots$, is complete since every τ_n can be expressed by means of it.)

§3. The above method was given in the writers thesis.[3] It is based on certain results of J. v. Neumann, in particular the relations $T^{**} = T$ and $\mathfrak{N}^* = [\mathfrak{R}]^{\perp}$.[4] Of course, one should point out that there has existed for a long time the method of E. Schmidt for solving an infinite system of equations in an infinite number of unknowns, which gives the same result.[5] However the Schmidt method does not explicitly use the adjoint and in order to apply it, it is first necessary to translate the given operational equation into an infinite system of linear equations. Our present method is immediately applicable to function spaces. Its basis is not the geometrical methods of Schmidt but the structure of operators as explained in §4 below.

In our present discussion, we have assumed certain properties of function spaces and orthonormal sets. For proofs of these the reader is referred to *M. H. Stone*, Chapter 1 or *F. J. Murray* (2), Chapter 2. In the present paper, all convergence for functions should be understood as convergence in the mean. We also assume the result of Theorem I of the writer's thesis without explicit mention.

§4. A linear set $\mathfrak{A}$ is defined as a set of functions, such that if f and g are in $\mathfrak{A}$, then $af + bg$ is in $\mathfrak{A}$ for any two complex numbers, a and b. A closed linear set $\mathfrak{M}$, is a linear set which contains all its limit functions with respect to the norm, i.e. if f is such that $\lim_{\alpha\to\infty} \| f - f_\alpha \| = 0$, for a sequence of functions $f_1, f_2, \cdots$ from $\mathfrak{M}$, then f is in $\mathfrak{M}$. With a closed linear set $\mathfrak{M}$, we can also consider $\mathfrak{M}^{\perp}$ the set of functions g such that $(f, g) = 0$, for all f in $\mathfrak{M}$. $\mathfrak{M}^{\perp}$ is a closed linear set and if f is an arbitrary function in L_2, f can be expressed as the sum of a function of $\mathfrak{M}$ and a function of $\mathfrak{M}^{\perp}$, i.e. $f = f' + f''$ where f' is in $\mathfrak{M}$ and f'' is in $\mathfrak{M}^{\perp}$. Clearly $(\mathfrak{M}^{\perp})^{\perp} = \mathfrak{M}$.

We may now attempt our justification in the above terms. We have already mentioned $\mathfrak{N}$, the set of functions such that $Tf = 0$. We have similarly $\mathfrak{N}^*$, the set of g's such that $T^*g = 0$. Let $\mathfrak{R}$ denote the set of functions g which result when T is applied, i.e. if g is in $\mathfrak{R}$, then $g = Tf$ for some f. Similarly $\mathfrak{R}^*$ is the set of T^*f's. In general, $\mathfrak{R}$ and $\mathfrak{R}^*$ are linear sets but not closed. However we may add their limit functions, to obtain closed linear sets $[\mathfrak{R}]$ and $[\mathfrak{R}^*]$. These are such that $\mathfrak{N}^{\perp} = [\mathfrak{R}^*]$ and $\mathfrak{N}^{*\perp} = [\mathfrak{R}]$. $\mathfrak{N}$ and $\mathfrak{N}^*$ are always closed.

If f is in $\mathfrak{L}_2$, then $f = f' + f''$ where f' is in $\mathfrak{N}$ and f'' is in $\mathfrak{N}^{\perp} = [\mathfrak{R}^*]$. f' in $\mathfrak{N}$ implies $Tf' = 0$ and hence $Tf = Tf''$. Thus if for a given g, there is an f such that $Tf = g$, then there is also an f'' in $[\mathfrak{R}^*]$, such that $Tf'' = g$. Now the $\varphi_1, \varphi_2, \cdots$ have been chosen so that $[\mathfrak{R}^*]$, such that $Tf'' = g$. Now the $\varphi_1, \varphi_2, \cdots$

[3] Cf. F. J. Murray (1), Theorems I and II. We have slightly modified the procedure, so that our present method is a step shorter in an actual calculation.

[4] The notation is explained in §4 below. Cf. J. v. Neumann, Satz 2 and 7.

[5] Cf. E. Schmidt or G. Kowalewski.

have been chosen so that $[\mathfrak{R}^*]$ is precisely the set of functions $f = \sum_{\alpha=1}^{\infty} x_\alpha \varphi_\alpha$, where $\sum |x_\alpha|^2 < \infty$. Thus if there is a solution for $Tf = g$, there is one in the form assumed in our method.

Note that while T maps $[\mathfrak{R}^*]$ on $\mathfrak{R}$ and $\mathfrak{N}$ on (0), T^* maps $[\mathfrak{R}]$ on $\mathfrak{R}^*$ and $\mathfrak{N}^*$ on (0).

We next use the fact that $T^{**} = T$. This is equivalent to the following statement: $Tf = g$ if and only if for every h,

$$(g, h) = (f, T^*h). \tag{8}$$

We consider this for an $f = \sum_{\alpha=1}^{\infty} x_\alpha \varphi_\alpha$ in $[\mathfrak{R}]$ and a g in $[\mathfrak{R}]$.

Now (8) is linear and continuous in h. Thus we need only consider it for a complete orthonormal set, $\chi_1, \chi_2, \cdots$,

$$(g, \chi_n) = (f, T^*\chi_n). \tag{8'}$$

Furthermore if $T^*\chi_n$, is linearly dependent on $T^*\chi_1, \cdots, T^*\chi_{n-1}$, this n may be omitted among the equations (8'). For if $T^*\chi_n = \sum_{\alpha=1}^{n-1} d_{n,\alpha} T^*\chi_\alpha$, then $T^*(\chi_n - \sum_{\alpha=1}^{n-1} d_{n,\alpha}\chi_\alpha) = 0$. Then $\chi_n - \sum_{\alpha=1}^{n-1} d_{n,\alpha}\chi_\alpha$ is in $\mathfrak{N}^*$ and since g is in $[\mathfrak{R}] = \mathfrak{N}^{*\perp}$, $(g, \chi_n - \sum_{\alpha=1}^{n-1} d_{n,\alpha}\chi_\alpha) = 0$. We have also $(f, T^*\chi_n - \sum_{\alpha=1}^{n-1} d_{n,\alpha} T^*\chi_\alpha) = 0$ and these two results yield $(g, \chi_n) - (f, T^*\chi_n) = \sum_{\alpha=1}^{n-1} d_{n,\alpha}((g, \chi_\alpha) - (f, T^*\chi_\alpha))$. Hence a linear dependence relation among the $T^*\chi_1, T^*\chi_2, \cdots$ yields a corresponding relation among the equations (8'). Thus we may confine our attention to the equations (8') for which the $T^*\chi_1, T^*\chi_2, \cdots$, are linearly independent, i.e.

$$(g, \chi_{n'}) = (f, T^*\chi_{n'}). \tag{8''}$$

The orthonormalization process on the series, $T^*\chi_{1'}, T^*\chi_{2'}, \cdots$ yields $\varphi_n = \sum_{\alpha=1}^{n} c_{n,\alpha} T^*\chi_{\alpha'}$ and $\omega_n = \sum_{\alpha=1}^{n} c_{n,\alpha}\chi_{\alpha'}$. Since $c_{n,n} \neq 0$, these equations show that (8'') is equivalent to

$$(g, \omega_n) = (f, \varphi_n). \tag{8'''}$$

Thus for f in $[\mathfrak{R}^*]$, g in $[\mathfrak{R}]$, $Tf = g$ is equivalent to (8''').

Now g in $[\mathfrak{R}]$, is of course equivalent to condition (a) in §2. This condition is necessary for $Tf = g$, since the latter implies that g is in $\mathfrak{R}$.

Condition (b) is a necessary and sufficient condition that $f = \sum_{\alpha=1}^{\infty} (g, \omega_\alpha)\varphi_\alpha$ should exist. If there is an f such that $Tf = g$, we know from the above that there is an f in $[\mathfrak{R}^*]$ such that $Tf = g$. $Tf = g$ implies (8''') for this latter f and hence $f = \sum_{\alpha=1}^{\infty} (f, \varphi_\alpha)\varphi_\alpha = \sum_{\alpha=1}^{\infty} (g, \omega_\alpha)\varphi_\alpha$. It follows that condition (b) must hold. Hence condition (b) is also necessary if there is an f such that $Tf = g$.

Together (a) and (b) are also sufficient. For (a) yields that g is in $[\mathfrak{R}]$ and (b) shows that $f = \sum_{\alpha=1}^{\infty} x_\alpha \varphi_\alpha$ exists where $x_\alpha = (f, \varphi_\alpha) = (g, \omega_\alpha)$. Thus we have (8''') and $Tf = g$. Thus (a) and (b) imply that there is an f such that $Tf = g$.

§5. In the preceding discussion, we have supposed that the set $\chi_1, \chi_2, \cdots$ is complete, but it is enough if $\chi_1, \chi_2, \cdots$ span $[\mathfrak{R}]$, i.e. $[\mathfrak{R}]$ consists of functions in the form $\sum_{\alpha=1}^{\infty} x_\alpha \varphi_\alpha$ where $\sum_{\alpha=1}^{\infty} |x_\alpha|^2 < \infty$. We have also assumed that T is bounded, i.e. there exists a constant C such that for all f in L_2, $\| Tf \| \leq C \| f \|$.

The last restriction may be considerably weakened if an extra precaution is taken. First we introduce the notion of the "graph" of an operator.[6] We form the space $\mathfrak{L}_2 \oplus \mathfrak{L}_2$, consisting of pairs $\{f, g\}$ of functions of $\mathfrak{L}_2$. The pairs are to be treated as two dimensional vectors with functions as components. The "graph" of a linear operator, T, is the linear set of pairs, $\{f, Tf\}$ in $\mathfrak{L}_2 \oplus \mathfrak{L}_2$. T is "closed" if the graph is closed in $\mathfrak{L}_2 \oplus \mathfrak{L}_2$, i.e. if the graph contains every limit pair $\{f, g\}$ of the pairs in it. The domain of T is the set of f's for which Tf can be formed and is in $\mathfrak{L}_2$. The domain is "dense" if every function of $\mathfrak{L}_2$ is a limit of functions in the domain. We wish to consider those operators which are linear, closed and have domain dense, abreviated l.c.d.d.

The l.c.d.d. operators contain those bounded operators which are defined for all f in $\mathfrak{L}_2$. Our method is applicable to l.c.d.d. operators for g in $[\mathfrak{R}]$, provided the pairs $\{\chi_n, T^*\chi_n\}$ span the graph of T^*, i.e. there is no $g \neq 0$ such that $(g, \chi_n) + (T^*g, T^*\chi_n) = 0$ for every n. (If there is such a g, we would add it to the χ_n's. The orthogonality of the χ_n's is unessential.) The restriction on the χ_n's may even be weakened to the following: There is no $g \neq 0$ and in $[\mathfrak{R}]$, such that $(g, \chi_n) + (T^*g, T^*\chi_n) = 0$ for every n. Condition (a) must hold however in the form "g is in $[\mathfrak{R}]$."

§6. We illustrate the above with a numerical example. We let $K(x, y) = \exp(\lambda^2 xy)$ where $\lambda = 1/2\pi$. We consider the equation

$$Tf = \int_0^{2\pi} \exp(\lambda^2 xy) f(x)\, dx = \lambda^{\frac{1}{2}}$$

and prove that there is no $f(x)$ in $\mathfrak{L}_2$, which satisfies it.

A number of preliminary remarks are necessary. First of all, it is clear that $T^* = T$. One can show also that $\mathfrak{N} = (0)$. (The following heuristic argument shows how this can be proven. Suppose $Tf = \int_0^{2\pi} \exp(\lambda^2 xy) f(x) dx = 0$. Then $0 = \frac{d}{dy} Tf = T(\lambda^2 x f(x))$ or $T(xf(x)) = 0$. This can be used to show that $T(p(x)f(x)) = 0$ for every polynomial $p(x)$. But $p(x)$ can be made to approximate $f(x)$ in the mean as closely as desired, so this yields $T|f|^2 = 0$. The latter is possible only if $f = 0$. These facts also give an alternate way of proving that there is no f such that $Tf = 1$. For differentiating the last equation yields $T(xf) = 0$ and hence $xf = 0$ and $f = 0$. The last is inconsistent with $Tf = 1$. However, this does not illustrate the method described.) $\mathfrak{N} = (0)$ and $T^* = T$ yield $\mathfrak{N}^* = (0)$ and $[\mathfrak{R}] = [\mathfrak{R}^*] = \mathfrak{L}_2$. Hence every g in $\mathfrak{L}_2$ is in $[\mathfrak{R}]$ and condition (a) is always fulfilled.

We take for $\chi_1, \chi_2, \cdots$ the trigonometrical series, i.e. $\chi_1 = \lambda^{\frac{1}{2}}$, $\chi_{2k} = (1/\pi)^{\frac{1}{2}} \sin kx$, $\chi_{2k+1} = (1/\pi)^{\frac{1}{2}} \cos kx$. (There are two reasons for this choice. In the first place $T^*\chi_n$ is relatively easy to calculate and in the second, our given $g =$

[6] The "graph" was introduced in J. v. Neumann. (Cf. loc. cit. p. 299, 5 or F. J. Murray (2) Def. 1, p. 31.)

$\chi_1 = \lambda^{\frac{1}{2}}$ is orthogonal to all but one χ_n and consequently the formula for (g, ω_n) is simpler.) Let $f_n = T^*\chi_n$. If we let $\rho(u) = (e^u - 1)/u$, $\sigma_k = 1/k^2$, we have

$$
\begin{aligned}
f_1 &= \lambda^{-\frac{1}{2}}\rho(\lambda y), \\
f_{2k} &= -2^{\frac{1}{2}}\lambda^{\frac{1}{2}}k^{-1}(\lambda y)(1 + \sigma_k\lambda^2(\lambda y)^2)^{-1}\rho(\lambda y), \\
f_{2k+1} &= 2^{\frac{1}{2}}\lambda^{\frac{1}{2}}k^{-2}(\lambda y)^2(1 + \sigma_k\lambda^2(\lambda y)^2)^{-1}\rho(\lambda y).
\end{aligned} \tag{9}
$$

Now let $[h(u)] = \int_0^1 h(u)\rho^2(u)\,du$. (Note the new limits of integration.) Suppose $i = 2k + r$, $j = 2l + s$, where $r = 0$ or 1, $s = 0$ or 1. Then for $i \neq 1$ and $j \neq 1$, we have

$$
\begin{aligned}
&(f_1, f_1) = \lambda^{-2}[1] \\
&(f_1, f_i) = (f_i, f_1) = -(-1)^r\lambda^{r-1}k^{-1-r}2^{\frac{1}{2}}[u^{1+r}(1 + \sigma_k\lambda^2u^2)^{-1}] \\
&(f_i, f_j) = (-1)^{r+s}\lambda^{r+s}2k^{-1-r}l^{-1-s}[u^{2+r+s}(1 + \sigma_k\lambda^2u^2)^{-1}(1 + \sigma_l\lambda^2u^2)^{-1}]
\end{aligned} \tag{10}
$$

To obtain $\varphi_1, \varphi_2, \cdots$, we must orthonormalize the $f_1, f_2, \cdots$. No f_n is linearly dependent on $f_1, \cdots, f_{n-1}$. For $f_n = \sum_{\alpha=1}^{n-1} a_\alpha f_\alpha$ implies $T^*(\chi_n - \sum_{\alpha=1}^{n-1} a_\alpha\chi_\alpha) = 0$. Since $\mathfrak{N}^* = (0)$, this implies $\chi_n - \sum_{\alpha=1}^{n-1} a_\alpha\chi_\alpha = 0$, which is impossible since the $\chi_1, \chi_2, \cdots$ are linearly independent. Thus the first step in the orthonormalization process does not affect $f_1, f_2, \cdots$ and the second yields

$$
\varphi_n(x) = \begin{vmatrix} f_1(x), & \cdots, f_n(x) \\ (f_1, f_1), & \cdots, (f_1, f_n) \\ \cdots & \cdots \\ (f_{n-1}, f_1), & \cdots, (f_{n-1}, f_n) \end{vmatrix} \div |A_{n-1}A_n|^{\frac{1}{2}} \tag{11}
$$

where A_n is the determinant,

$$
A_n = |(f_i, f_j)|_{i,j=1,\cdots,n}.
$$

Consequently

$$
\omega_n = \begin{vmatrix} \chi_1, & \cdots, \chi_n \\ (f_1, f_1), & \cdots, (f_1, f_n) \\ \cdots & \cdots \\ (f_{n-1}, f_1), & \cdots, (f_{n-1}, f_n) \end{vmatrix} \div |A_{n-1}A_n|^{\frac{1}{2}} \tag{12}
$$

Now our given g is χ_1. Since $(\chi_1, \chi_1) = 1$, $(\chi_1, \chi_n) = 0$ for $n \neq 1$, we have

$$
(g, \omega_n) = B_n \div |A_{n-1}A_n|^{\frac{1}{2}}
$$

where B_n is the determinant,

$$
B_n = |(f_i, f_j)|_{i=1,\cdots,n-1;\ j=2,\cdots,n}.
$$

Let $c_n = B_n^2 \div |A_{n-1}A_n|$. Condition (b) for the equation $Tf = \chi_1$ is then $\sum_{\alpha=1}^{\infty} c_\alpha < \infty$. We shall prove in what follows that $\sum_{\alpha=1}^{\infty} c_\alpha = \infty$ and hence that there is no f in $\mathfrak{L}_2$ such that $Tf = \chi_1$.

§7. We must therefore evaluate c_n. To do this, we must express the determinants A_n, A_{n-1}, B_n as products.

Referring to (10), we see that (f_i, f_j) has certain factors outside the square brackets. These can be factored from our determinants. The result is

$$
\begin{aligned}
A_{2s} &= 2^{2s-1}\lambda^{2s-4}A^I_{2s} \div ((s-1)!)^6 \cdot s^2,\\
A_{2s-1} &= 2^{2s-2}\lambda^{2s-4}A^I_{2s-1} \div ((s-1)!)^6,\\
B_{2s} &= 2^{2s-1}\lambda^{2s-3}B^I_{2s} \div ((s-1)!)^6 s,\\
B_{2s-1} &= 2^{2s-1}\lambda^{2s-4}B^I_{2s-1} \div ((s-2)!)^6(s-1)^4,
\end{aligned} \tag{13}
$$

where $A^I_{2s} = |a_{i,j}|$ with

$$a_{i,j} = [u^{2+r+s}(1+\sigma_k\lambda^2u^2)^{-1}(1+\sigma_l\lambda^2u^2)^{-1}].$$

($\sigma_0 = 0$ in this formula.) A^I_{2s-1} is obtained from A^I_{2s} by omitting the $2s$ row and $2s$ column, B^I_{2s} is obtained from A^I_{2s} by omitting the $2s$ row and 1st column, B^I_{2s-1} is obtained from A^I_{2s} by omitting the $2s$ and $2s-1$ rows and the 1st and $2s$ columns.

Note that

$$
\begin{aligned}
B^2_{2s} \div A_{n-1}A_n &= \lambda^2(B^{I2}_{2s} \div A^I_{n-1}A^I_n),\\
B^2_{2s-1} \div A_{n-2}A_{n-1} &= \lambda^2(B^{I2}_{2s-1} \div A^I_{2s-2}A^I_{2s-1}).
\end{aligned} \tag{14}
$$

(When one extracts factors from rows and columns, symmetrically, only the factors from the first row or first column will not be cancelled out in the expression for c_n.)

In the new determinants, adjacent rows or columns are nearly equal, so we will use a version of the Hankel process involving both rows and columns.[7] Specifically we proceed as follows for A_{2s}.

We note first that

$$(1+\sigma_k\lambda^2u^2)^{-1} - (1+\sigma_2\lambda^2u^2)^{-1} = \lambda^2u^2(\sigma_2-\sigma_k)(1+\sigma_k\lambda^2u^2)^{-1}(1+\sigma_2\lambda^2u^2)^{-1}.$$

We subtract the 2nd row from the 4th, 6th, $\cdots$, $2s$ rows and the 3rd row from the 5th, $\cdots$, $2s-1$ rows. We then factor out $(\prod_{p=3}^{s-1}(\sigma_2-\sigma_p)^2)(\sigma_2-\sigma_s)$. We then perform precisely the same operations on the columns.

Next, we subtract the 4th row from the 6th, $\cdots$, $2s$ rows and the 5th row from the 7th, $\cdots$ $2s-1$ rows. Then we factor out $(\prod_{p=4}^{s-1}(\sigma_3-\sigma_p)^2)(\sigma_3-\sigma_s)$. We repeat this for the columns.

The above is repeated for every pair of rows and columns up to the $2s-2$, $2s-1$ pair.

The final result is

$$
\begin{aligned}
A^I_{2s} &= (\prod_{q=2}^{s-1}(\prod_{p=q+1}^{s-1}(\sigma_q-\sigma_p)^4))\prod_{q=2}^{s-1}(\sigma_q-\sigma_s)^2A^{II}_{2s},\\
A^I_{2s-1} &= (\prod_{q=2}^{s-1}\prod_{p=q+1}^{s-1}(\sigma_q-\sigma_p)^4)A^{II}_{2s-1},\\
B^I_{2s} &= (\prod_{q=2}^{s-1}(\prod_{p=q+1}^{s-1}(\sigma_q-\sigma_p)^4)\prod_{q=2}^{s-1}(\sigma_q-\sigma_s)B^{II}_{2s},\\
B^I_{2s-1} &= (\prod_{q=2}^{s-2}(\prod_{p=q+1}^{s-1}(\sigma_q-\sigma_p)^4))\prod_{q=2}^{s-2}(\sigma_q-\sigma_{s-1})B^{II}_{2s-1},
\end{aligned}
$$

[7] Cf. H. Hankel or T. Muir or G. Kowalewski.

where $A^{II}_{2s} = (b_{i,j})$ with $b_{1,1} = [1]$ and for $i, j \neq 1$,

$$b_{1,s} = b_{i,1} = \lambda^{2(k-1)}[u^{i-1}(\prod_{p=1}^{k}(1+\sigma_p\lambda^2u^2))^{-1}]$$

$$b_{i,j} = \lambda^{2(k-1)+2(l-1)}[u^{i+j-2}(\prod_{p=1}^{k}(1+\sigma_p\lambda^2u^2)\prod_{q=1}^{l}(1+\sigma_q\lambda^2u^2))^{-1}].$$

The determinants A^{II}_{2s-1}, B^{II}_{2s}, B^{II}_{2s-1} are obtained by omitting the appropriate rows and columns of A^{II}_{2s}.

Factoring out the λ factor yields

$$A^{II}_{2s} = \lambda^{4s^2-4}A^{III}_{2s}, \qquad B^{II}_{2s} = \lambda^{4s^2-2s-2}B^{III}_{2s}$$

$$A^{II}_{2s-1} = \lambda^{4s^2-4s}A^{III}_{2s-1}, \qquad B^{II}_{2s-1} = \lambda^{4s^2-6s}B^{III}_{2s-1}$$

where $A^{III}_{2s} = |c_{i,j}|$ with

$$c_{i,j} = [u^{i+j-2}(\prod_{p=1}^{k}(1+\sigma_p\lambda^2u^2)\prod_{q=1}^{l}(1+\sigma_q\lambda^2u^2))^{-1}]$$

The first row has contributed no factor other than one to these factorizations and thus we have

$$B^{I^2}_n/|A^I_{n-1}A^I_n| = B^{III^2}_n/|A^{III}_{n-1}A^{III}_n|, \qquad n = 2s-1, 2s. \tag{15}$$

We now make the approximation

$$c_{i,j} = (e-1)^2 \div (i+j)\prod_{p=1}^{k}(1+\sigma_p\lambda^2)\prod_{q=1}^{l}(1+\sigma_q\lambda^2).$$

(This is based on a consideration of the integral described by the square brackets.) Substituting and factoring yields

$$A^{III}_{2s} = (e-1)^{4s}A^{IV}_{2s} \div (\prod_{k=1}^{s-1}\prod_{p=1}^{k}(1+\sigma_p\lambda^2)^4)\prod_{p=1}^{s}(1+\sigma_p\lambda^2)^2$$

$$A^{III}_{2s-1} = (e-1)^{4s-2}A^{IV}_{2s-1} \div \prod_{k=1}^{s-1}\prod_{p=1}^{k}(1+\sigma_p\lambda^2)^4$$

$$B^{III}_{2s} = (e-1)^{4s-2}B^{IV}_{2s} \div \prod_{k=1}^{s-1}\prod_{p=1}^{k}(1+\sigma_p\lambda^2)^4\prod_{p=1}^{s}(1+\sigma_p\lambda^2)$$

$$B^{III}_{2s-1} = (e-1)^{4s-4}B^{IV}_{2s-1} \div \prod_{k=1}^{s-2}\prod_{p=1}^{k}(1+\sigma_p\lambda^2)^4\prod_{p=1}^{s-1}(1+\sigma_p\lambda^2)^3$$

where $A^{IV}_{2s} = (1/(i+j))_{i,j=1,\cdots,2s}$.

We now have

$$B^{III^2}_n/A^{III}_{n-1}\cdot A^{III}_n = B^{IV^2}_n/A^{IV}_{n-1}A^{IV}_n, \qquad n = 2, 3, \cdots.$$

(The distinction between odd and even has disappeared) and using (14) and (15)

$$B^2_n/|A_{n-1}A_n| = \lambda^2(e-1)^{-2}B^{IV^2}_n/|A^{IV}_{n-1}A^{IV}_n|. \tag{16}$$

Now A^{IV}_n is a type of determinant, evaluated by Ligowski, who used the following method.[8] The following formula is due to Cauchy

$$|1/(\alpha_i-\beta_j)|_{i,j=1,\cdots,n}$$

$$= \prod_{i=1}^{n}\prod_{k=i+1}^{n}(\alpha_k-\alpha_i)\prod_{j=1}^{n}\prod_{l=j+1}^{n}(\beta_j-\beta_l) \div \prod_{i=1}^{n}\prod_{j=1}^{n}(\alpha_i-\beta_j).$$

[8] Cf. W. Ligowski or T. Muir.

If we let $\alpha_i = n + i, \beta_j = n - j$, this determinant becomes A_n^{IV} and thus

$$A_n^{IV} = (\prod_{i=1}^{n-1} i!)^3 n! \div \prod_{i=1}^{n} (n + i)!$$

Now $B_n^{IV} = |1/(i + j + 1)|_{i,j=1,\cdots,n-1}$ and for this we substitute $\alpha_i = n + i, \beta_j = n - 1 - j$, in the Cauchy formula and obtain

$$B_n^{IV} = (\prod_{j=1}^{n-2} j!)^3 (n - 1)! n! \div \prod_{i=1}^{n-1} (n + i)!$$

Also

$$A_{n-1}^{IV} = (\prod_{j=1}^{n-2} j!)^3 (n - 1)! \div \prod_{i=1}^{n-1} (n - 1 + i)!$$

Consequently, we obtain

$$B_n^{IV^2} / A_{n-1}^{IV} A_n^{IV} = 2 \cdot n^3, \qquad n = 2, 3, \cdots$$

(16) then yields

$$c_n = B_n^2 / A_{n-1} A_n = \lambda^2 (e - 1)^{-2} \cdot 2n^3, \qquad n = 2, 3, \cdots$$

and clearly $\sum_{n=1}^{\infty} c_n = \infty$.

Thus there is no f such that $Tf = \chi_1$. A similar argument holds for $Tf = \chi_n$ for $n = 2, 3, \cdots$. (If this result itself were of interest, it could be obtained by a differentiation argument similar to that given in §6 for $Tf = \chi_1$.)

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BIBLIOGRAPHY

M. Bocher. "Introduction to Higher Algebra." The MacMillan Co. New York, N. Y. (1929 Edition.)
H. Hankel. "Über eine besondere Classe der Symmetrischen Determinanten." Dissertation, Gottigen, (1861).
G. Kowalewski. "Determinantentheorie." Leipzig (1909), pp. 407–455.
W. Ligowski. Archiv. d. Math. u. Physik., Vol. XXXVI (1861), pp. 181–185.
T. Muir. "Theory of Determinants." Vol. III (1920), MacMillan and Co., London, pp. 311–316.
F. J. Murray. (1). Trans. Amer. Math. Soc. Vol. 37, (1935), pp. 301–338.
(2). "Linear Transformations in Hilbert Space," Annals of Math. Studies, No. 4, Princeton, New Jersey, (1941).
E. Schmidt. Rend. Circ. Mat. Palermo. Vol. XXV, (1908), pp. 53–77.
M. H. Stone. "Linear Transformations in Hilbert Space." Amer. Math. Soc. Colloquium Publications, Vol. XV, (1932), New York, N. Y.
J. v. Neumann. Annals of Math (2), Vol. 33, (1932), pp. 294–310.

CHARACTERIZATION OF THE SYSTEM OF EXTREMALS OF VARIATION PROBLEM OF HIGHER ORDER IN THE PLANE

By Meyer Karlin

PART I

CHAPTER I

Section 1

Introduction

The present paper represents a solution of the following inverse problem in the calculus of variations:

Given a family of ∞^4 curves in the 2-dimensional (x, y) space as represented by the differential equation of the fourth order:

$$y^{IV} = F(x, y, y', y'', y''') \tag{1.1}$$

to determine whether these curves can be identified with the totality of extremals of some variation problem:

$$\int \varphi(x, y, y', y'')\, dx = \min. \tag{1.2}$$

i.e. whether there exist such functions $\varphi(x, y, y', y'')$ whose Euler-Lagrange equations are identified with (1.1), and in the affirmative case to find all the corresponding functions.

This problem was proposed to the author by Professor J. Douglas, who also made a number of helpful suggestions during the course of work.

As will be shown later, we prove that (1.1) represents a totality of extremals if and only if $F(x, y, y', y'', y''')$ is of the form $A + By''' + Cy'''^2$ where A, B and C are functions of x, y, y' and y'' obeying a system of six differential equations which we call (I), (II), (III), (IV) (V) and (VI).

When these differential equations are satisfied, we can then actually find the corresponding function φ and we note that this function φ is essentially unique, meaning up to the multiplication by a constant factor and the addition of an arbitrary total derivative of a function of x, y, y' with respect to x.

In part II we obtain some results for the corresponding third order problem where ∞^6 curves are defined by a differential equation or the sixth order.

History

Section 2

In 1894 G. Darboux[1] solved the simplest type of the inverse problem in the calculus of variations, namely the one concerning a doubly infinite system of curves represented by the differential equation:

$$y'' = F(x, y, y') \tag{1.3}$$

[1] G. Darboux, *Lecons Sur La Theorie Generale des Surfaces*. Paris, 1834, §§604, 605

He showed that any such family represents the totality of extremals of some variation problem:

$$\int \varphi(x, y, y')\, dx = \min. \tag{1.4}$$

It is well known that in this respect the simplest type of problem is unique. As we turn to equations involving higher derivatives than the second, or when we deal with more than one dependent or independent variable, the situation is entirely different. The differential equations must obey some specific conditions in order that the families of curves represented by them be identified with a totality of extremals.

In 1837 Jacobi[2] proved an important theorem in connection with the Euler-Lagrange equation of any function $\varphi(x, y, y', y'', \cdots, y^{(n)})$ of one dependent and one independent variable, involving differentials of any order.

We know that the Euler-Lagrange differential equation:

$$E(\varphi) \equiv \frac{\partial \varphi}{\partial y} - \frac{d}{dx}\frac{\partial \varphi}{\partial y'} + \frac{d^2}{dx^2}\frac{\partial \varphi}{\partial y''} - \cdots + \cdots + (-1)^n \frac{d^n}{dx^n}\frac{\partial \varphi}{\partial y^{(n)}} = 0 \tag{1.5}$$

corresponding to a 2-dimensional problem of any order n

$$\int \varphi(x, y, y', y'', \cdots, y^{(n)})\, dx = \min. \tag{1.6}$$

is of even order.

In a fundamental memoir in the calculus of variations, Jacobi proved that if we expand the Euler-Lagrange equation (1.5), and form its variational expression, linear in δy,

$$\delta E \equiv E_y \delta y + E_{y'} \delta y' + \cdots + E_{y^{(2n)}} \delta y^{(2n)} \tag{1.7}$$

then this expression will be self-adjoint. That is, designating for brevity $\delta y \equiv U$ and rewriting δE as a linear homogeneous expression in U

$$\delta E \equiv \mathfrak{L}(U) = E_0 U + E_1 U' + \cdots + E_{2n} U^{(2n)} \tag{1.8}$$

$$\text{where } E_j \equiv E_{y^{(j)}} \quad (j = 0, 1, \cdots, 2n)$$

we will find that this expression in U is self-adjoint. The adjoint of $\mathfrak{L}(U)$ is by definition:

$$\mathfrak{M}(U) \equiv E_0 U - (E_1 U)' + (E_2 U)'' - \cdots + \cdots + (-1)^k (E_k U)^{(k)} + \cdots + (E_{2n} U)^{(2n)} \tag{1.9}$$

where the differentiations indicated are total differentiations with respect to x.

Jacobi's theorem, in fact, states that as a linear homogeneous expression in U

$$\mathfrak{M}(U) \equiv \mathfrak{L}(U) \tag{1.10}$$

[2] Collected works, Vol. 4, *Zur Theorie der Variationsrechnung und der Differentialgleichungen.*

In 1897 A. Hirsch[3] proved the converse, namely that if $W(x, y, y', \cdots, y^{(2n)})$ is any differential expression whose variation δW (a linear differential expression in δy) is self-adjoint, then W must be identifiable with the Euler-Lagrange expression E, corresponding to some integrand φ as defined by (1.5). This result was later extended to expressions involving partial differential equations of several dependent variables.

In 1940 J. Douglas[4] gave a complete solution of the most important case of a system of differential equations, namely the 3-dimensional case where the given family consists of ∞^4 curves defined by differential equations of form:

$$y'' = F(x, y, z, y', z'), \qquad z'' = G(x, y, z, y', z') \tag{1.11}$$

In 1907 E. Kasner[5] showed that the Euler-Lagrange expression for the plane problem of second order (which is the problem dealt with in this paper):

$$\int \varphi(x, y, y', y'')\, dx = \min. \tag{1.12}$$

is of the form:

$$y^{\mathrm{IV}} = A + By''' + Cy'''^2 \tag{1.13}$$

where A, B, C involve only x, y, y', y''. On the basis of this, Kasner deduced an important geometric property of the extremals connected with a plane problem of second order, namely:

The extremals connected with any plane problem of the second order, that is with any integral of type (1.12), form a quadruply infinite system of curves such that ∞^1 are determined by a given element of curvature. The locus of the centers of conics which osculate these ∞^1 curves at their common point is a conic tangent to the given element.

The present paper is based upon the application of the condition of self-adjointness to the equation (1.13) multiplied by an integrating factor $\varphi_{y''y''}$ which was cancelled out in the derivation of (1.13). For convenience we replace $\varphi_{y''y''}$ by M, which represents an unknown function of x, y, y', y'' whose expression must be determined to find φ.

Chapter II

We start with the expression:

$$W = MA + MBy''' + MCy'''^2 + My^{\mathrm{IV}} \tag{2.1}$$ [6]

[3] *Über eine Charakteristische Eigenschaft die Differentialgleichungen der Variationsrechnung*, Math. Annalen, Vol. 49. (1897), pp. 49–72

[4] Two articles in Proceedings of the National Academy of Sciences and a longer paper *Solution of the Inverse Problem of the Calculus of Variations*, Trans. Amer. Math. Soc., Vol. 50. (1941) No. 1 pp. 71–128

[5] *Systems of Extremals in the Calculus of Variations*, Bull. Amer. Math. Soc. (1907).

[6] For convenience sake we took $y^{\mathrm{IV}} = -(A + By''' + Cy'''^2)$ (so that in applications we have to regard A, B, C with a minus sign).

where M, A, B, C are functions of x, y, y', y'', M representing $\varphi_{y''y''}$. By Hirsch's theorem we know that $W = 0$, represents the expression of the Euler-Lagrange differential equation for some $\int \varphi(x, y, y', y'')\,dx = \min$ if and only if the differential expression:

$$\delta W = W_y\delta y + W_{y'}\delta y' + W_{y''}\delta y'' + W_{y'''}\delta y''' + W_{y^{IV}}\delta y^{IV} \tag{2.2}$$

is self-adjoint.

Writing $\delta y \equiv U$ we obtain for δW a linear homogeneous expression in U

$$\mathfrak{L}(U) = W_y U + W_{y'}U' + W_{y''}U'' + W_{y'''}U''' + W_{y^{IV}}U^{IV} \tag{2.3}$$

whose adjoint is:

$$\mathfrak{M}(U) \equiv W_y U - (W_{y'}U)' + (W_{y''}U)'' - (W_{y'''}U)''' + (W_{y^{IV}}U)^{IV} \tag{2.4}$$

where the symbols of differentiation represent total differentiation with respect to x.

Hence the necessary and sufficient condition that $W = 0$ represents a quadruply infinite family of extremals of some plane variatonal problem of the second order is, that

$$\mathfrak{L}(U) \equiv \mathfrak{M}(U) \tag{2.5}$$

Expanding the terms of $\mathfrak{M}(U)$ and equating the coefficients of the various derivatives of U to the corresponding coefficients in $\mathfrak{L}(U)$ we get four differential equations only two of which are independent[7]

$$2\frac{dM}{dx} = M(B + 2Cy''') \tag{2.6}$$

$$\frac{d^2}{dx^2}M(B + 2Cy''') - 2\frac{d}{dx}M(A_{y''} + B_{y''}y''' + C_{y''}y'''^2) + 2M(A_{y'} + B_{y'}y''' + C_{y'}y'''^2) = 0 \tag{2.7}$$

Thus (2.5) reduces itself to a system of two differential equations (2.6) and (2.7) which form the necessary and sufficient conditions that (1.13) represent a quadruply infinite family of extremals.

Chapter III

Section 1

We are now ready to derive our fundamental system. Since (2.6) and (2.7) are identities in y''' we expand $\frac{dM}{dx} = M_x + M_y y' + M_{y'}y'' + M_{y''}y'''$ and

[7] This result was to be expected for A. Hirsch in his paper proved that self-adjointness leads to n conditions and not $2n$.

equate the coefficient of y''' and the terms free of y''' and obtain the following results:

$$M_{y''} = MC \tag{3.1}$$

$$M_x + M_y y' + M_{y'} y'' = \tfrac{1}{2} MB \tag{3.2}$$

We solve these two equations by putting $L \equiv \operatorname{Log} M$ (3.1) and (3.2) can be written as

$$\delta(x, y, y', y'') \equiv L_{y''} = C \tag{3.1^a}$$

$$L_x + L_y y' + L_{y'} y'' = \tfrac{1}{2} B \tag{3.2^a}$$

In order that L and therefore M be integrable, we must know that the order of differentiation in mixed partial derivatives be immaterial. We, therefore, differentiate (3.2^a) with respect to y'' and apply the results of differentiating (3.1^a) with respect to x, y and y' and we obtain $C_x + C_y y' + C_{y'} y'' + L_{y'} = \tfrac{1}{2} B_{y''}$

$$\gamma(x, y, y', y'') \equiv L_{y'} = \tfrac{1}{2} B_{y''} - C_x - C_y y' - C_{y'} y'' \tag{3.3}$$

Differentiating (3.3) with respect to y'' and (3.1^a) with respect to y' and stating that $L_{y'y''} = L_{y''y'}$ we obtain

$$\gamma_{y''} = \delta_{y'} \tag{I}$$

or

$$\tfrac{1}{2} B_{y''y''} - C_{xy''} - C_{yy''} y' - C_{y'y''} y'' - 2C_{y'} = 0 \tag{3.4}$$

Differentiating now (3.2^a) with respect to y' we get

$$\gamma_x + L_y + \gamma_y y' + \gamma_{y'} y'' = \tfrac{1}{2} B_{y'}$$

or

$$\beta(x, y, y', y'') \equiv L_y = \tfrac{1}{2} B_{y'} - \gamma_x - \gamma_y y' - \gamma_{y'} y'' \tag{3.5}$$

Combining (3.5), (3.3) and (3.2^a)

$$\alpha(x, y, y', y'') \equiv L_x = \tfrac{1}{2} B - \beta y' - \gamma y'' \tag{3.6}$$

Differentiating (3.5) with respect to y' and (3.3) with respect to y we get

$$\beta_{y'} = \gamma_y \tag{II}$$

Similarly from (3.5) and (3.1^a) we get the result

$$\beta_{y''} = \delta_y \tag{III}$$

Differentiating (3.2^a) with respect to y we obtain in view of (3.5)

$$\beta_x + \beta_y y' + \beta_{y'} y'' = \tfrac{1}{2} B_y \tag{IV}$$

We have shown that $\gamma_{y''} = \delta_{y'}$; $\gamma_y = \beta_{y'}$; $\beta_{y''} = \delta_y$. It can be easily shown that the three other relations between the mixed partials hold. Differentiating (3.6) with respect to y'', y', y we get the following three relations

$$\alpha_{y''} = \tfrac{1}{2}B_{y''} - \beta_{y''}y' - \gamma_{y''}y'' - \gamma \tag{3.6a}$$

$$\alpha_{y'} = \tfrac{1}{2}B_{y'} - \beta_{y'}y' - \gamma_{y'}y'' - \beta \tag{3.6b}$$

$$\alpha_{y} = \tfrac{1}{2}B_{y} - \beta_{y}y' - \gamma_{y}y'' \tag{3.6c}$$

Using the fact that $\gamma = \frac{1}{2}B_{y''} - C_x - C_y y' - C_{y'}y''$ and that $\beta_{y''} = \delta_y \equiv C_y$, $\gamma_{y''} = \delta_{y'} \equiv C_{y'}$ we see immediately from (3.6a) that $\alpha_{y''} = C_x \equiv \delta_x$. Again, remembering that $\beta = \frac{1}{2}B_{y'} - \gamma_x - \gamma_y y' - \gamma_{y'}y''$ and that $\beta_{y'} = \gamma_y$, we obtain from (3.6b) that $\alpha_{y'} = \gamma_x$. Equation (IV) shows that in view of the fact that $\beta_{y'} = \gamma_y$ we get from (3.6c) $\alpha_y = \beta_x$. Thus we see that the symbolic matrix

$$\begin{Vmatrix} \dfrac{\partial}{\partial x}, & \dfrac{\partial}{\partial y}, & \dfrac{\partial}{\partial y'}, & \dfrac{\partial}{\partial y''} \\ \alpha, & \beta, & \gamma, & \delta \end{Vmatrix} = 0$$

Hence we see that if (I), (II), (III), and (IV) hold we can integrate

$$L = \int L_x\,dx + L_y\,dy + L_{y'}\,dy' + L_{y''}\,dy'' = \int \alpha\,dx + \beta\,dy + \gamma\,dy' + \delta\,dy'' \tag{3.7}$$

where α, β, γ and δ are defined by (3.6), (3.5) and (3.3) and (3.1a) respectively, and we know that the integrand will be an exact differential.

To obtain M we put

$$M = e^{L}\,^{8} \tag{3.8}$$

In examining (2.7) we observe that in virtue of (2.6), the equation can be written so as to have M as a factor and not contain any derivatives of M. Carrying out the indicated differentiations and bearing in mind (2.6) as well as the fact that

$$A + By''' + Cy'''^2 + y^{IV} = 0$$

we will get an expression which is an identity in y'''. It will be seen that the coefficient of y'''^3 will vanish identically. While the other coefficients yield the following results

$$-B_{y''y''} + 2C_{xy''} + 2C_{yy''}y' + 2C_{y'y''}y'' + 4C_{y'} = 0 \tag{3.9}$$

This is the same as (I)

$$\begin{aligned}(V)\quad \Phi(x,y,y',y'') \equiv\ & 3B_{y'} + \tfrac{3}{2}BB_{y''} - BC_x - BC_y y' - BC_{y'}y'' + \tfrac{1}{2}B^2C + B_xC \\ & + B_yCy' + B_{y'}Cy'' - 2AC^2 + 2C_{xx} + 4C_{xy}y' + 4C_{xy'}y'' + 2C_{yy}y'^2 \\ & + 4C_{yy'}y'y'' + 2C_{y'y'}y'^2 + 2C_yy'' - 2AC_{y''} - 4A_{y''}C - 2A_{y''y''} = 0\end{aligned}$$

[8] Since L is determined up to an additive constant, M is determined up to a constant factor.

$$\text{(VI)}\quad \Psi(x, y, y', y'') \equiv \tfrac{1}{4}B^3 + \tfrac{3}{2}BB_x + \tfrac{1}{2}BB_y y' + \tfrac{3}{2}BB_{y'}y'' - ABC + AB_{y'} - A_{y'}B + B_{xx} + 2B_{xy}y' + 2B_{xy'}y'' + B_{yy}y'^2 + 2B_{yy'}y'y'' + B_{y'y'}y''^2 + B_y y'' - 4AC_x - 4AC_y y' - 4AC_{y'}y'' - 2A_x C - 2A_y Cy' - 2A_{y'}Cy'' - 2A_{xy'} - 2A_{yy'}y' - 2A_{y'y'}y'' + 2A_{y'} = 0$$

(I), (II), (III), (IV), (V) and (VI) represent the fundamental system of differential equations, representing the necessary and sufficient conditions in order that (1.13) represent an ∞^4 family of extremals.

Section 2

We can derive our fundamental system directly from the Euler-Lagrange equation without the use of Hirsch's theorem. We know that the ∞^4 system of extremals of the variation problem (1.2) must satisfy

$$\begin{aligned} E(\varphi) \equiv \varphi_y - \frac{d}{dx}\varphi_{y'} + \frac{d^2}{dx^2}\varphi_{y''} \equiv{}& \varphi_y - \varphi_{xy'} - \varphi_{yy'}y' - \varphi_{y'y'}y'' \\ &+ \varphi_{xxy''} + \varphi_{yyy''}y'^2 + \varphi_{y'y'y''}y''^2 + \varphi_{yy''}y'' + 2\varphi_{xyy''}y' \\ &+ 2\varphi_{xy'y''}y'' + 2\varphi_{yy'y''}y'y'' + y'''(2\varphi_{xy''y''} + 2\varphi_{yy''y''}y' \\ &+ 2\varphi_{y'y''y''}y'') + \varphi_{y''y''y''}y'''^3 + \varphi_{y''y''}y^{\mathrm{IV}} = 0 \end{aligned} \tag{3.10}$$

In order that our system of extremals be identifiable with the ∞^4 system of curves

$$y^{\mathrm{IV}} = -(A + By''' + Cy'''^2) \tag{3.11}$$

Where A, B and C are given functions of x, y, y' and y'' it is necessary that the result of substituting this value of y^{IV} in (3.10) yield an identity in y'''

$$\begin{aligned} E(\varphi, A, B, C) \equiv{}& \varphi_y - \varphi_{xy'} - \varphi_{yy'}y' - \varphi_{y'y'}y'' + \varphi_{xxy''} \\ &+ \varphi_{yyy''}y'^2 + \varphi_{y'y'y''}y''^2 + \varphi_{yy''}y'' + 2\varphi_{xyy''}y' + 2\varphi_{xy'y''}y'' \\ &+ 2\varphi_{yy'y''}y'y'' - A\varphi_{y''y''} + (2\varphi_{xy''y''} + 2\varphi_{yy''y''}y' + 2\varphi_{y'y''y''}y'' \\ &- B\varphi_{y''y''})y''' + (\varphi_{y''y''y''} - C\varphi_{y''y''})y'''^2 \equiv 0 \end{aligned} \tag{3.12}$$

Equating the coefficients of the various powers of y''' to zero identically, we get from the coefficient of y'''^2

$$\varphi_{y''y''y''} - C\varphi_{y''y''} \equiv 0 \qquad \text{(compare (3.1))} \tag{3.13}$$

The coefficient of y''' yields

$$2\varphi_{xy''y''} + 2\varphi_{yy''y''}y' + 2\varphi_{y'y''y''}y'' - B\varphi_{y''y''} \equiv 0 \qquad \text{(compare 3.2)} \tag{3.14}$$

Equations (3.13) and (3.14) enable us to solve for $\varphi_{y''y''}$ in terms of B and C. Denoting $\operatorname{Log} \varphi_{y''y''} \equiv L$ we can proceed as above to derive the necessary conditions (I), (II), (III), and (IV) and obtain

$$\varphi_{y''y''} = e^{\int \alpha dx + \beta dy + \gamma dy' + \lambda dy''} \tag{3.15}$$

Since (3.12) is an identity, we can differentiate the whole expression with respect to any variable. Keeping in mind (3.13) and (3.14) it will be found that

$$\frac{d}{dx}\frac{\partial}{\partial y''}E(\varphi, A, B, C) - \frac{\partial}{\partial y'}E(\varphi, A, B, C) = \tfrac{1}{2}\Psi(x, y, y', y'') + \tfrac{1}{2}\Phi(x, y, y', y'')y''' \tag{3.16}$$

This proves the necessity of (V) and (VI).

Section 3

Suppose we are given an ∞^4 system of curves

$$y^{IV} = -(A + By''' + Cy'''^2) \tag{A}$$

where A, B, C are given functions of x, y, y' and y'' obeying our fundamental system (I), (II), (III), (IV), (V), and (VI). We will proceed to show that these curves can be identified with the totality of extremals of some variation problem of second order in the plane, and actually obtain the function to be minimized.

Using equations (3.7) and (3.8) we can obtain a function M

$$M = e^{\int \alpha dx + \beta dy + \gamma dy' + \lambda dy''} \tag{3.17}$$

where the integrand is an exact differential in virtue of (I), (II), (III), and (IV).

We next solve the differential equation

$$\varphi_{y''y''} = M \tag{3.18}$$

This gives φ determined up to an arbitrary linear function of y''. Suppose $\bar{\varphi}(x, y, y', y'')$ is a particular solution of (3.18). In virtue of (I), (II), (III), and (IV) we see that equations (3.13) and (3.14) are satisfied.

Hence we obtain:

$$\begin{aligned} E(\bar{\varphi}) = {} & \bar{\varphi}_y - \bar{\varphi}_{xy'} - \bar{\varphi}_{yy'}y' - \bar{\varphi}_{y'y'}y'' + \bar{\varphi}_{xxy''} + \bar{\varphi}_{yyy''}y'^2 + \bar{\varphi}_{y'y'y''}y''^2 \\ & + \bar{\varphi}_{yy''}y'' + 2\bar{\varphi}_{xyy''}y' + 2\bar{\varphi}_{xy'y''}y'' + 2\bar{\varphi}_{yy'y''}y'y'' - A\bar{\varphi}_{y''y''} \end{aligned} \tag{3.19}$$

That is $E(\bar{\varphi})$, is independent of y'''.

Furthermore, differentiating (3.19) twice with respect to y'' we obtain

$$\begin{aligned} \frac{\partial^2}{\partial y''^2}E(\bar{\varphi}) = {} & 3\bar{\varphi}_{yy''y''} + 3\bar{\varphi}_{xy'y''y''} + 3\bar{\varphi}_{yy'y''y''}y' + 3\bar{\varphi}_{y'y'y''y''}y'' \\ & + \bar{\varphi}_{xxy''y''y''} + \bar{\varphi}_{yyy''y''y''}y'^2 + \bar{\varphi}_{y'y'y''y''y''}y''^2 + \bar{\varphi}_{yy''y''y''}y'' \\ & + 2\bar{\varphi}_{xyy''y''y''}y' + 2\bar{\varphi}_{xy'y''y''y''}y'' + 2\bar{\varphi}_{yy'y''y''y''}y'y'' \\ & - A_{y''y''}\bar{\varphi}_{y''y''} - 2\bar{\varphi}_{y''y''y''}A_{y''} - A\bar{\varphi}_{y''y''y''y''} \end{aligned} \tag{3.20}$$

It can be shown that in virtue of (V) we will have

$$\frac{\partial^2}{\partial y''^2}E(\bar{\varphi}) = 0 \tag{3.20^a}$$

Thus, while we cannot guarantee that $E(\bar{\varphi}) = 0$, we do know that it is at least of the form

$$E(\bar{\varphi}) = \xi(x, y, y') + \eta(x, y, y')y'' \tag{3.21}$$

Any other solution of (3.18) can be written as:

$$\varphi = \bar{\varphi} + \lambda(x, y, y')y'' + \mu(x, y, y') \tag{3.22}$$

We must be able to obtain λ and μ so that

$$E(\varphi) = 0$$

Expanding $E(\varphi)$ we get

$$E(\varphi) = E(\bar{\varphi}) + \lambda_y y'' + \mu_y - \frac{d}{dx}(\lambda_{y'} y'' + \mu_{y'}) + \frac{d^2\lambda}{dx^2} \tag{3.23}$$

In order that $E(\varphi) = 0$ we must have the right member equal zero. Expanding and equating the coefficients of various powers of y'' to zero, we obtain the following set of conditions on λ and μ

$$P \equiv 2\lambda_y + \lambda_{xy'} + \lambda_{yy'}y' - \mu_{y'y'} + \eta(x, y, y') = 0 \tag{3.24}$$

$$Q \equiv \mu_y - \mu_{xy'} - \mu_{yy'}y' + \lambda_{xx} + 2\lambda_{xy}y' + \lambda_{yy}y'^2 + \xi(x, y, y') = 0 \tag{3.25}$$

Differentiating (3.25) with respect to y' we get:

$$\frac{\partial Q}{\partial y'} = -\mu_{xy'y'} - \mu_{yy'y'}y' + \lambda_{xy'} + 2\lambda_{xyy'}y' + 2\lambda_{xy} + \lambda_{yyy'}y'^2 + 2\lambda_{yy}y' + \xi_{y'} \tag{3.26}$$

Differentiating (3.24) with respect to x and with respect to y, we obtain

$$\begin{cases} \dfrac{\partial P}{\partial x} = 2\lambda_{xy} + \lambda_{xxy'} + \lambda_{xyy'}y' - \mu_{xy'y'} + \eta_x \\ \dfrac{\partial P}{\partial y} = 2\lambda_{yy} + \lambda_{xyy'} + \lambda_{yyy'}y' - \mu_{yy'y'} + \eta_y \end{cases} \tag{3.27}$$

Equations (3.26) and (3.27) help us to eliminate λ and μ obtain a set of necessary conditions on ξ and η

$$\frac{\partial Q}{\partial y'} - \frac{\partial P}{\partial x} - \frac{\partial P}{\partial y}y' = \xi_{y'} - \eta_x - \eta_y y' \tag{3.28}$$

If λ and μ are to exist at all satisfying (3.24) and (3.25) we, thus, must have the following relation in ξ and η satisfied

$$\xi_{y'} - \eta_x - \eta_y y' = 0 \tag{3.29}$$

In view of (3.21) this equation can be rewritten as

$$\frac{\partial}{\partial y'}E(\bar{\varphi}) - \frac{d}{dx}\frac{\partial}{\partial y''}E(\bar{\varphi}) = 0 \tag{3.29*}$$

Going back to (3.16), we can see that, in virtue of the fact that A, B and C obey equations (V) and (VI), this condition holds.

We have thus shown that in virtue of the fact that the function A, B and C of our ∞^4 system of curves (A), obey our fundamental system (I), (II), (III), (IV), (V) and (VI), we can obtain $\bar{\varphi}$ such that $E(\bar{\varphi})$ is of the form (3.21) and that furthermore it is possible to find φ such that $E(\varphi) = 0$ if we can solve (3.24) and (3.25) for λ and μ. We have shown, furthermore, that the necessary condition (3.29) holds. We now choose for $\lambda(x, y, y')$ any particular function $\bar{\lambda}$

$$\bar{\lambda} = \bar{\lambda}(x, y, y') \tag{3.30}$$

and solve $P = 0$ for μ

$$\mu_{y'y'} = 2\bar{\lambda}_y + \bar{\lambda}_{xy'} + \bar{\lambda}_{yy'}y' + \eta(x, y, y') \tag{3.31}$$

This determines μ up to a linear function in y'. Choose some particular solution as $\bar{\mu}$

$$\bar{\mu} = \bar{\mu}(x, y, y') \tag{3.32}$$

Our choice of $\bar{\lambda}$ and $\bar{\mu}$ guarantees us the following:

$$\begin{cases} \overline{P}(x, y, y') \equiv P(\eta, \bar{\lambda}, \bar{\mu}) = 0 \\ \dfrac{\partial \bar{Q}}{\partial y'} - \dfrac{\partial \overline{P}}{\partial x} - \dfrac{\partial \overline{P}}{\partial y}y' = 0 \end{cases} \tag{3.33}$$

from (3.29) and (3.28), which can be rewritten as

$$\overline{P} = 0, \qquad \frac{\partial \bar{Q}}{\partial y'} = 0 \tag{3.33^a}$$

or

$$\bar{Q}(x, y, y') \equiv Q(\xi, \bar{\lambda}, \bar{\mu}) = \tau(x, y) \tag{3.33^b}$$

We next try to choose a different μ to make $Q = 0$. Equation (3.31) shows us that any μ is of the form

$$\mu = \bar{\mu} + a(x, y)y' + b(x, y) \tag{3.34}$$

For any such μ we have

$$P = \overline{P} = 0, \qquad Q = \bar{Q} + b_y - a_x = \tau(x, y) + b_y - a_x \tag{3.35}$$

Hence if we choose $a(x, y)$ and $b(x, y)$ subject to the condition:

$$a_x - b_y = \tau(x, y) \tag{3.36}$$

We will have $P = 0$ and $Q = 0$. It is well known that (3.36) can be solved. Taking any particular solution a_0 and b_0 we will have:

$$\lambda_0 = \bar{\lambda}, \qquad \mu_0 = \bar{\mu} + a_0(x, y)y' + b_0(x, y) \tag{3.37}$$

These two functions yield:

$$P(\eta, \lambda_0, \mu_0) = 0 \quad \text{and} \quad Q(\xi, \lambda_0, \mu_0) = 0 \tag{3.38}$$

And hence if we put:

$$\varphi_0 = \bar{\varphi} + \lambda_0(x, y, y')y'' + \mu_0(x, y, y') \tag{3.39}$$

we will have

$$E(\varphi_0) = 0 \tag{3.40}$$

We will now show that our solution is essentially unique. Suppose that there exists another solution of (3.18) $\varphi_1 = \varphi_1(x, y, y', y'')$ and such that

$$E(\varphi_1) = 0 \tag{3.41}$$

we will show that

$$\varphi_1 = \varphi_0 + \frac{d}{dx}\nu(x, y, y') \tag{3.42}$$

In the first place we know from (3.22) that:

$$\varphi_1 = \bar{\varphi} + \lambda_1(x, y, y')y'' + \mu_1(x, y, y') \tag{3.43}$$

Thus we have to show that there exists a function $\nu(x, y, y')$. Such that

$$\psi(x, y, y') \equiv \lambda_0 - \lambda_1 = \nu_{y'}, \qquad \zeta(x, y, y') \equiv \mu_0 - \mu_1 = \nu_x + \nu_y y' \tag{3.42a}$$

We can easily solve the first of equations (3.42a). Suppose $\bar{\nu}(x, y, y')$ is a particular solution of $\nu_{y'} = \psi$. Then we have:

$$\lambda_1 = \lambda_0 + \bar{\nu}_{y'}, \qquad \mu_1 = \mu_0 + \zeta(x, y, y') \tag{3.42b}$$

Now λ_1 and μ_1 have to satisfy $P = 0$ and $Q = 0$. Substituting λ_1 and μ_1 in (3.24) and (3.25) we get

$$\begin{cases} \bar{\nu}_{yy'} + \bar{\nu}_{xy'y'} + \bar{\nu}_{yy'y'}y' - \zeta_{y'y'} = 0 \\ \zeta_y - \zeta_{xy'} - \zeta_{yy'}y' + \bar{\nu}_{xxy'} + 2\bar{\nu}_{xyy'}y' + \bar{\nu}_{yyy'}y'^2 = 0 \end{cases} \tag{3.44}$$

The first of these states that:

$$\frac{\partial^2}{\partial y'^2}(\zeta - \bar{\nu}_y y' - \bar{\nu}_x) = 0 \tag{3.45}$$

or

$$\zeta = \bar{\nu}_x + \nu_y y' + \rho(x, y)y' + \sigma(x, y) \tag{3.45a}$$

Substituting this into the second of equations (3.44), we get the equation in ρ and σ

$$\sigma_y - \rho_x = 0 \tag{3.45b}$$

We now choose a different solution of $\nu_{y'} = \psi$

$$\nu = \bar{\nu} + N(x, y) \tag{3.46}$$

in such a fashion that (3.42a) will be satisfied. This is possible if

$$N_x = -\sigma, \qquad \text{and} \qquad N_y = -\rho \tag{3.47}$$

Equation (3.45[b]) shows us that such an $N(x, y)$ exists. Hence we get:

$$\lambda_1 = \lambda_0 + \nu_{y'}, \qquad \mu_1 = \mu_0 + \nu_x + \nu_y y' \tag{3.48}$$

or

$$\varphi_1 = \varphi_0 + \nu_x + \nu_y y' + \nu_{y'} y'' = \varphi_0 + \frac{d}{dx}\nu(x, y, y') \tag{3.49}$$

Chapter IV

Examples

We will now give some examples of extremal and non-extremal families. We will take some simplified cases.

1) Suppose $B = C = 0$ i.e. our differential equation is of the form

$$y^{IV} + A(x, y, y', y'') = 0 \tag{4.1}$$

Equations (3.1[a]), (3.3), (3.5) and (3.6) yield

$$L = \text{constant or } M = K \text{ (constant)} \tag{4.2}$$

(V) yields:

$$-2A_{y''y''} = 0 \quad \text{or} \quad A = f(x, y, y') + g(x, y, y')y'' \tag{4.3}$$

(VI) yields:

$$-2A_{xy''} - 2A_{yy''}y' - 2A_{y'y''}y'' + 2A_{y'} = 0 \tag{4.4}$$

or in view of (4.3)

$$f_{y'} = g_x + g_y y' \tag{4.5}$$

We can choose g but f will be determined up to a function independent of y'. We will take special cases

a) $$f = -24, \qquad g = 0, \qquad y^{IV} = 24$$

we get that

$$y = x^4 + ax^3 + bx^2 + cx + d \tag{4.6}$$

represents an extremal family.

As a check we will find φ and show that $E(\varphi) \equiv 0$. Now $\varphi_{y''y''} \equiv M = k$ (constant)

$$\varphi = \int_0^{y''} dy'' \int_0^{y''} \kappa\, dy'' + y'''\lambda + \mu = \kappa y''^2 + \lambda(x, y, y')y'' + \mu(x, y, y') \tag{4.7}$$

$$E(\varphi) = \varphi_y - \frac{d}{dx}\varphi_{y'} + \frac{d^2}{dx^2}\varphi_{y''}$$

$$\begin{aligned} &= \lambda_y \cdot y'' + \mu_y - \frac{d}{dx}(\lambda_{y'}y'' + \mu_{y'}) + \frac{d^2}{dx^2}(2\kappa y'' + \lambda) \\ &= (2\lambda_y + \lambda_{xy'} + \lambda_{yy'}y' - \mu_{y'y'})y'' + \mu_y - \mu_{xy'} - \mu_{yy'}y' \\ &\qquad + \lambda_{xx} + 2\lambda_{xy}y' + \lambda_{yy}y'^2 + 2\kappa y^{IV} \end{aligned} \tag{4.8}$$

This means that $E(\varphi) \equiv 0$ if:

(4.9) $$2\lambda_y + \lambda_{xy'} + \lambda_{yy'}y' - \mu_{y'y'} = 0$$

(4.10) $$\mu_y - \mu_{xy'} - \mu_{yy'}y' + \lambda_{xx} + 2\lambda_{xy}y' + \lambda_{yy}y'^2 + 48k = 0$$

These two formulas are exactly (3.24) and (3.25) and we obtain a particular solution by taking $\lambda = 0$, $\mu = -48ky$, we thus have the result:

I) An ∞^4 family of quartics whose leading coefficient is constant is an extremal family for the integral of $ky''^2 - 48ky + \frac{d}{dx}\nu(x, y, y')$

b) $$f = 0, \quad g = 1, \quad y^{IV} = -y''$$

(4.11) $$y = c \sin x + d \cos x + ax + b$$

is an extremal family for the same type of an integral only with different λ and μ

(4.9^b) $2\lambda_y + \lambda_{xy'} + \lambda_{yy'} - \mu_{y'y'} - 2\kappa = 0$

(4.10^b) $\mu_y - \mu_{xy'} - \mu_{yy'}y' + \lambda_{xx} + 2\lambda_{xy}y' + \lambda_{yy}y'^2 = 0$

} This combines into $\mu = 0$, $\lambda = \kappa y$

II) An ∞^4 family of sine curves built upon straight lines is an extremal family for the integral of $ky''^2 + k_2y'' + \frac{d}{dx}\nu(x, y, y')$

c) $$f = -y, \quad g = 0, \quad y^{IV} = y$$

(4.12) $$y = c_1e^x + c_2e^{-x} + c_3 \sin x + c_4 \cos x$$

is also an extremal family for an integral of the same type

$$\varphi = \kappa(y''^2 - y^2) + \frac{d}{dx}\nu(x, y, y')$$

2) $$A = B = 0$$

Following the method outlined above we get

(4.13) $y^{IV} + C(x, y, y', y'')y''^2 = 0$ is an ∞^4 system of extremals for which:

(4.14) $L_{y''} = C, \quad L_{y'} = -C_x - C_{y'}y'', \quad L_y = 0, \quad L_x = C_xy'' + C_{y'}y''^2;$

(4.15) $C_y = 0$ or $C = C(x, y', y'')$

(4.16) $C_{xx} + 2C_{xy'}y'' + C_{y'y'}y''^2 = 0$

(4.17) $2C_{y'} + C_{xy''} + C_{y'y''}y'' = 0$

special cases

a) $$C = 1, \quad y^{IV} = -y''^2$$

(4.18) $$y = \frac{(x + d)^2}{2} \log (x + d) + ax^2 + bx + c$$

is an extremal family. To get φ we observe that in this case

(4.19) $L = y'' + K_1$, $M = K_2 e^{y''}$, and we obtain by the above method:

(4.20) $$\varphi = \kappa e^{y''} + \frac{d}{dx}\nu(x, y, y')$$

b) $$c = x, \quad \text{or} \quad y^{IV} = -xy'''^2$$

(4.21) $$y = \left(\frac{x^2}{d^2} - 2d + 1\right)\tan^{-1}\frac{x}{d} - x\log(d^2 + x^2) + ax^2 + bx + c$$

is an extremal family whose φ is represented by

(4.22) $$\varphi = \kappa\frac{e^{xy''-y'}}{x^2} + \frac{d}{dx}\nu(x, y, y')$$

3) We get a very interesting result if we take A, B, C to be constants. We get from (V) and (VI) the two formulas

(4.23) $$\begin{cases} B(\frac{1}{4}B^2 - AC) = 0 \\ C(\frac{1}{2}B^2 - 2AC) = 0 \end{cases}$$

Since we examined already the case $B = C = 0$, we will now examine the possibility

(4.23a) $$B^2 - 4AC = 0$$

Hence if A, B and C are constants y^{IV} must be the square of a linear expression in y'''.

Thus we get the non-extremal family which has the algebraic form (1.13) of extremal families

(4.24) $$y^{IV} = y^{III}$$

or

(4.24a) $$y = Ke^x + ax^2 + bx + c$$

In case (4.23a) holds, we can rewrite

$$y^{IV} = (ay''' + b)^2,$$

(4.24) $$y = -\frac{1}{a^4}(ax + \kappa)^2\left[\frac{\log(ax + \kappa)}{2} - \frac{1}{4}\right] - \frac{b}{a}x^3 + lx^2 + mx + n$$

where k, l, m, n are the arbitrary constants. We can rewrite it as:

(4.24a) $$y = -\frac{1}{2a^4}(ax + \kappa)^2\log(ax + \kappa) - \frac{b}{a}x^3 + l_1x^2 + m_1x + n_1$$

It is readily seen that (4.18) is a special case of this.

To solve for φ we observe as above that in this case

$$Ly''' \equiv C \equiv a^2, \qquad Lx = \tfrac{1}{2}B = ab, \qquad Ly' = 0, \qquad Ly = 0 \tag{4.25}$$

$$L = a^2y'' + abx + k_1, \qquad M = K_2 e^{a^2y'' + abx} \tag{4.26}$$

$$\varphi = \frac{1}{a^4} K_2 e^{a^2y'' + abx} + \frac{d}{dx}\nu(x, y, y'). \tag{4.27}$$

4) If we take $A = C = 0$ or $y^{IV} = By'''$ we get a more complicated situation. We will merely point out that B must be of the form:

$$B = 2\frac{y'' + \rho'(x)y + \sigma'(x) - \overline{\rho(x)}^2 y - \rho(x)\sigma(x)}{y' + \rho(x)y + \sigma(x)}$$

where ρ and σ are functions of x only and are such that (V) must be satisfied. *Special case*:($\rho = 0$, $\sigma = 0$ will satisfy (V). Thus we get the extremal family $B = \dfrac{2y''}{y'}$

$$y^{IV} = -\frac{2y''y'''}{y'} \quad \text{or} \quad y'y^{IV} + 2y''y''' = 0. \tag{4.29}$$

Where φ will be found to be of the form:

$$\varphi = \kappa y'y''^2 + \frac{d}{dx}\nu(x, y, y') \tag{4.30}$$

The differential equation (4.29) can be solved by putting $y' = z$

$$zz''' + 2z'z'' \tag{4.29^a}$$

Consider z as independent variable and $z' = q$ as dependent variable we obtain dividing through by $q(\neq 0)$

$$zq\frac{d^2q}{dz^2} + z\left(\frac{dq}{dz}\right)^2 + q\frac{dq}{dz} = -q\frac{dq}{dz} \tag{4.30}$$

We will finally get an expression for x and y involving z as a parameter

$$\begin{cases} x = \dfrac{1}{a}\sqrt{z(az - b)} + \dfrac{b}{2a\sqrt{a}}\log\left(\sqrt{z(az - b)} + z\sqrt{a} + \dfrac{b}{2\sqrt{a}}\right) + c \\ y = \dfrac{z}{2a}\sqrt{z(az - b)} + \dfrac{3a}{4c}x + d \end{cases} \tag{4.31}$$

where a, b, c, d are arbitrary constants.

PART II

Chapter V

Section 1

We shall now try to investigate the ∞^6 family of curves in the plane as expressed by the differential equation of the sixth order

$$y^{VI} = f(x, y, y', y'', y'''. y^{IV}, y^{V}) \tag{5.1}$$

in order to determine whether these curves can be identified with the totality of extremals of some variation problem of third order:

$$\int \varphi(x, y, y', y'', y''')\, dx = \min. \tag{5.2}$$

This case represents much more detailed calculations so that we shall suffice ourselves with merely pointing out some of the necessary conditions obtained by essentially the same methods as for the case of ∞'' system of curves. The lengthy calculations are of necessity omitted.

The Euler-Lagrange equation for this case is

$$E(\varphi) = \varphi_y - \frac{d}{dx}\varphi_{y'} + \frac{d^2}{dx^2}\varphi_{y''} - \frac{d^3}{dx^3}\varphi_{y'''} \tag{5.3}$$

Now[9] φ_y, $\varphi_{y'}$, $\varphi_{y''}$, $\varphi_{y'''}$ are functions of x, y, y', y'', y''' and for an arbitrary function $g(x, y, y', y'', y''')$ we have

$$\frac{dg}{dx} = g_1 + g_2 \cdot y^{IV} \tag{5.4}$$

$$\begin{aligned}\frac{d^2 g}{dx^2} &= g_{11} + g_{12}\cdot y^{IV} + (g_{21} + g_{22} y^{IV}) y^{IV} + g_2 y^{V} \\ &= h_1 + h_2 y^{IV} + h_3 y^{IV^2} + h_4 y^{V}\end{aligned} \tag{5.5}$$

$$\begin{aligned}\frac{d^3 g}{dx^3} &= h_{11} + h_{12} y^{IV} + (h_{21} + h_{22} y^{IV}) y^{IV} + h_2 y^{V} + (h_{31} + h_{32} y^{IV}) y^{IV^2} \\ &\quad + 2h_3 y^{IV} y^{V} + (h_{41} + h_{42}\cdot y^{IV}) y^{V} + h_4 \cdot y^{VI} \\ &= \kappa_1(x, y, y', y'', y''') + \kappa_2(x, y, y', y'', y''')\cdot y^{IV} + \kappa_3(x, y, y', y'', y''') y^{IV} \\ &\quad + \kappa_4(x, y, y', y'', y''') y^{IV^2} + \kappa_5(x, y, y', y'', y''') y^{IV} y^{V} \\ &\quad + \kappa_6(x, y, y', y'', y''') y^{V} + \kappa_7(x, y, y', y'', y''') y^{VI}\end{aligned} \tag{5.6}$$

Applying this to (5.2) we see that:

$$\begin{aligned}E(\varphi) = A^{(1)} + B^{(1)} y^{IV} + C^{(1)} y^{IV^2} + D^{(1)} y^{IV^3} + E^{(2)} y^{V} + F^{(1)} y^{IV} y^{V} \\ - \varphi_{y'''y'''y'''} y^{IV} = 0\end{aligned} \tag{5.7}$$

[9] In deriving the formula (5.8) I am following the suggestion of the referee.

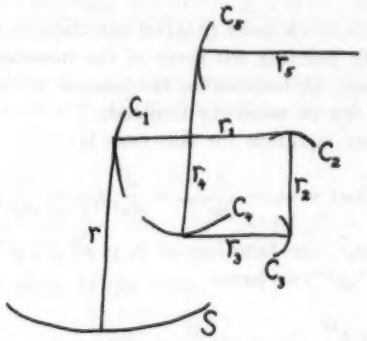
We thus obtain that the differential equation for the system of extremals for the variation problem of the third order in the plane is of the form:

$$y^{VI} = A + By^{IV} + cy^{IV^2} + Dy^{IV^3} + Ey^{V} + Fy^{IV}y^{V} \tag{5.8}$$

where A, B, C, D, E and F are functions of x, y, y', y'' and y'''.

This shows us that the ∞^6 curves which are identified with the totality of extremals must possess peculiar geometric properties. These properties may be stated in terms of the radii of second, third and fourth evolutes.

It is well known that r_n the radius of curvature of the n^{th} evolute is perpendicular to r_{n-1} so that starting with the original curve, the successive radii of curvature form a figure of something like this:



In view of this and the property of the evolute that

$$ds_n = dr_{n-1} \tag{5.9}$$

where ds_n refers to the differential of arc of the n^{th} evolute, we get the following formula:

$$r_n = \frac{ds_n}{d\theta} = \frac{ds_n}{dx} \Big/ \frac{dx}{d\theta} \tag{5.10}$$

now

$$\frac{d\theta}{dx} = \frac{d}{dx}\left(\tan^{-1}\frac{dy}{dx}\right) = \frac{\dfrac{d^2y}{dx^2}}{1 + (dy/dx)^2}$$

Hence we obtain the recursion formula:

$$r_n = \frac{dr_{n-1}}{dx} \cdot \frac{1 + y'^2}{y''} \tag{5.11}$$

now

$$r = \frac{(1 + y'^2)^{\frac{1}{2}}}{y''} \tag{5.12}$$

$$(5.13) \qquad r_1 = \frac{1+y'^2}{y''}\cdot\frac{dr}{dx} = \frac{(1+y'^2)^{\frac{1}{2}}(3y'y''^2 - y''' - y'^2y''')}{y''^3}$$

$$(5.14) \qquad r_2 = \frac{1+y'^2}{y''}\frac{dr_1}{dx} = a_1 + B_1 y^{IV}$$

Where $A_1 = A_1(y', y'', y''')$ and $B_1 = B_1(y', y'')$

$$(5.15) \qquad r_3 = \frac{1+y'^2}{y''}\frac{dr_2}{dx} = C_1 + D_1 y^{IV} + B_2 y^{V}$$

where C_1 and D_1 are functions of y', y'' and y''' and B_2 is a function of y' and y'' only

$$(5.16) \qquad r_4 = \frac{1+y'^2}{y''}\frac{dr_3}{dx} = E_1 + F_1 y^{IV} + G_1 y^{IV^2} + H_1 y^{V} + B_3 y^{VI}$$

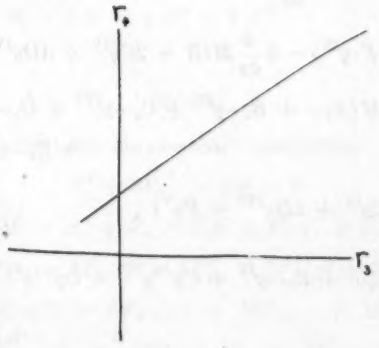
where E_1 to H, are functions of y', y'', y''', while B_3 is a function of y' and y''.

We are now ready to deduce our geometric properties. An extremal is determined when the initial values of $x, y, y', y', y''', y^{IV}$ and y^{V} are assigned. If only x, y, y', y'', y''' and y^{IV} are given, i.e., if we are given an element with fixed curvature and radii of curvature of first and second evolutes, then we have an ∞^1 of extremals having a fourth order contact. For this ∞^1 extremals we can deduce easily the locus of the fifth center of curvature.

Formula (5.15) shows that r_3 is linear in y^{V} and (5.16) shows that r_4 is linear in y^{V} and y^{VI}. Formula (5.8) reduces to a linear relation between y^{VI} and y^{V}. We thus get the relation:

$$(5.17) \qquad r_4 = \alpha + \beta r_3$$

where α and β are functions of x, y, y', y'', y''' and y^{IV} (or x, y, y', r, r_1 and r_2) and we get the theorem:



The extremals connected with any plane problem of the third order, that is with any integral of type (5.2), form an ∞^6 system of curves, such that ∞^1 are determined by a given element of fourth order. The locus of the fifth center of curvature of these curves is a straight line.

We call this property I.

We now allow y^{IV} to vary, and we obtain an ∞^2 of curves with fixed curvature and radius of curvature of first evolute. We will obtain a relation between three variable radii of curvature, second third and fourth.

We get from (5.14), (5.15), and (5.16)

$$\text{(5.18)} \qquad \begin{cases} y^{IV} = a + br_2 \\ y^{V} = c + dr_2 + er_3 \\ y^{VI} = f + gr_2 + hr_2^2 + jr_3 + kr_4 \end{cases}$$

where a to k are functions of y', y'' and y''' (or y', r and r_1). Equations (5.18) together with (5.8) yield:

$$\text{(5.19)} \qquad r_4 = G + Hr_2 + Ir_2^2 + Jr_2^3 + Kr_3 + Lr_2r_3$$

We can state property II in the following theorem:

The extremals connected with any plane problem of the third order that is with an integral of type (5.2) form an ∞^6 system of curves, such that an ∞^2 are determined by a given element of third order. The relation between the second, third and fourth centers of curvature being expressible by equation (5.19) which represents a ruled surface in an auxiliary r_2, r_3, r_4 space.

Section 2

To obtain the conditions on the coefficients A, B, C, D, E and F of equation (5.3) we follow the method outlined in chapter II. This yields a system of three differential equations:

$$\text{(5.20)} \qquad \frac{dM}{dx} = \tfrac{1}{3}M(E + F \cdot y^{IV})$$

$$\text{(5.21)} \qquad \begin{aligned} &5\frac{d^2}{dx^2}M(E + F\cdot y^{IV}) - 6\frac{d}{dx}M(B + 2Cy^{IV} + 3Dy^{IV^2} + Fy^{V}) \\ &\quad + 3M(A_{y'''} + B_{y'''}y^{IV} + C_{y'''}y^{IV^2} + D_{y'''}y^{IV^3} \\ &\qquad + E_{y'''}y^{V} + F_{y'''}y^{IV}y^{V}) = 0 \end{aligned}$$

$$\text{(5.22)} \qquad \begin{aligned} &\frac{d^3}{dx^3}M(B + 2Cy^{IV} + 3Dy^{IV^2} + Fy^{V}) \\ &- 3\frac{d^2}{dx^2}M(A_{y'''} + B_{y'''}y^{IV} + Cy'''y^{IV^2} + Dy'''y^{IV^3} \\ &\qquad + Ey'''y^{V} + F_{y'''}y^{IV}y^{V}) \\ &+ 5\frac{d}{dx}M(A_{y''} + B_{y''}y^{IV} + C_{y''}y^{IV^2} + D_{y''}y^{IV^3} + E_{y''}y^{V} + F_{y''}y^{IV}y^{V}) \\ &- 5M(A_{y'} + B_{y'}y^{IV} + C_{y'}y^{IV^2} + D_{y'}y^{IV^3} + E_{y'}y^{V} + F_{y'}y^{IV}y^{V}) \end{aligned}$$

We examine (5.20) as above and get that

$$L \equiv \int L_x\,dx + L_y\,dy + L_{y'}\,dy' + L_{y''}\,dy'' + L_{y'''}\,dy'''$$

(5.23)

$$= \frac{1}{3}\int \alpha\,dx + \beta\,dy + \gamma\,dy' + \delta\,dy'' + \eta\,dy'''$$

will be an exact differential, i.e.

(5.23a)
$$\begin{Vmatrix} \alpha & \beta & \gamma & \delta & \eta \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial y'} & \frac{\partial}{\partial y''} & \frac{\partial}{\partial y'''} \end{Vmatrix} = 0 \quad \text{if}$$

(I)
$$\frac{\partial \delta}{\partial y'''} = \frac{\partial \eta}{\partial y''}$$

(II)
$$\frac{\partial \gamma}{\partial y''} = \frac{\partial \delta}{\partial y'}$$

(III)
$$\frac{\partial \beta}{\partial y'} = \frac{\partial \gamma}{\partial y}$$

(IV)
$$\frac{\partial \alpha}{\partial y} = \frac{\partial \beta}{\partial x}$$

where:

(5.24) $$\eta = F$$

(5.25) $$\delta = E_{y'''} - F_x - F_{y'}y' - F_{y''}y'' - F_{y'''}y'''$$

(5.26) $$\gamma = E_{y''} - \delta_x - \delta_{y'}y' - \delta_{y''}y'' - \delta_{y'''}y'''$$

(5.27) $$\beta = E_{y'} - \gamma_x - \gamma_y y' - \gamma_{y'}y'' - \gamma_{y''}y'''$$

(5.28) $$\alpha = E - \beta \cdot y' - \gamma \cdot y'' - \delta \cdot y'''$$

To obtain M we put

(5.29) $$M = e^L$$

Equation (5.21) yields the following necessary conditions:

(V) $$F^2 + 3F_{y'''} - 9D = 0$$

(VI) $$2E_{y'''} + EF - 3C + F_x + F_y y' + F_{y'}y'' + F_{y''}y''' = 0$$

(VII)
$$\begin{aligned} &\tfrac{5}{3}EF^2 + 5EF_{y'''} + 5E_{y'''}F + 5FF_x + 5FF_y y' + 5FF_{y'}y'' \\ &\quad + 5FF_{y''}y''' + 5E_{y'''y'''} + 10F_{xy'''} + 10F_{yy'''}y' \\ &\quad + 10F_{y'y'''}y'' + 10F_{y''y'''}y''' + 5F_{y''} - 3CF \\ &\quad - 6DE - 9C_{y'''} - 18D_x - 18D_y y' - 18D_{y'}y'' - 18D_{y''}y''' = 0 \\ &\tfrac{5}{3}E^2F + 5EE_{y'''} + 5EF_x + 5EF_y y' + 5EF_{y'}y'' + 5EF_{y''}y''' + 5E_xF \\ &\quad + 5E_yFy' + 5E_{y'}Fy'' + 5E_{y''}Fy''' + 10E_{xy'''} + 10E_{yy'''}y' \end{aligned}$$

$$
\begin{aligned}
(\text{VIII}) \qquad & + 10E_{y'y'''}y'' + 10E_{y''y'''}y''' + 5E_{y''} + 5F_{xx} + 10F_{xy}y' \\
& + 10F_{xy'}y'' + 10F_{xy''}y''' + 5F_{yy}y'^2 + 10F_{yy'}y'y'' + 10F_{yy''}y'y''' \\
& + 5F_yy'' + 5F_{y'y'}y''^2 + 10F_{y'y''}y''y''' + 5F_{y''y''}y'''^2 \\
& - BF - 4CE - 3B_{y'''} - 12C_x - 12C_yy' - 12C_{y'}y'' - 12C_{y''}y''' = 0
\end{aligned}
$$

$$
\begin{aligned}
(\text{IX}) \qquad & \tfrac{5}{9}E^2 + 5EE_x + 5EE_yy' + 5EE_{y'}y'' + 5EE_{y''}y''' + 5E_{xx} \\
& + 10E_{xy}y' + 10E_{xy'}y'' + 10E_{xy''}y''' + 5E_{yy}y'^2 + 10E_{yy'}y'y'' \\
& + 10E_{yy''}y'y''' + 5E_yy'' + 5E_{y'y'}y''^2 + 10E_{y'y''}y''y''' \\
& + 5E_{y'}y''' + 5E_{y''y''}y'''^2 \\
& + AF - 2BE - 6B_x - 6B_yy' - 6B_{y'}y'' - 6B_{y''}y''' + 3A_{y'''} = 0
\end{aligned}
$$

Equation (5.22) when expanded consists of more than two thousand terms and we will therefore suffice with the partial results obtained which enable us to construct some extremal as well as non-extremal families of the algebraic form (5.8). We can see from equation (IX) that a necessary condition for the ∞^6 family

$$
y^{\text{VI}} = A(x, y, y', y'', y''') \tag{5.30}
$$

is

$$
A_{y'''} = C \tag{5.31}
$$

thus

$$
y^{\text{VI}} = y^{\text{III}} \tag{5.32}
$$

or

$$
y = ae^x + e^{-x/2}\left(b\cos\frac{\sqrt{3}}{2}x + c\sin\frac{\sqrt{3}}{2}x\right) + dx^2 + ex + f \tag{5.32^a}
$$

is an example of a non-extremal family while

$$
y^{\text{VI}} = k \qquad \text{(constant)} \tag{5.33}
$$

or

$$
y = x^6 + ax^5 + bx^4 + cx^3 + dx^2 + ex + f \tag{5.33^a}
$$

is an extremal family for the integral

$$
\int (y'''^2 + x^3y''')\,dx = \min. \tag{5.54}
$$

Note: We can easily see that, in general, the ∞^{2n} family of curves

$$
y = x^{2n} + \sum_{i=1}^{2n} a_i x^{2n-i} \tag{5.35}
$$

is an extremal family for the integral

$$\int (y^{(n)^2} + x^n y^{(n)})\, dx = \min. \tag{5.36}$$

Section 3

We wish to point out that property I deduced for the third order case can be extended to the n^{th} order case.

The Euler-Lagrange equation for the integral

$$\int \varphi(x, y, y', \cdots, y^{(n)})\, dx = \min. \tag{5.37}$$

is

$$E(\varphi) = \varphi_y - \frac{d}{dx}\varphi_{y'} + \frac{d^2}{dx^2}\varphi_{y''} - \cdots + \cdots + (-1)^n \frac{d^n}{dx^n}\varphi_{y^{(n)}} \tag{5.38}$$

We can easily see that $y^{(2n)}$ is the expansion of $E(\varphi)$ occurs only in the term $\frac{d^n}{dx^n}\varphi_{y^{(n)}}$ while y^{2n-1} will occur in the terms $\frac{d^n}{dx^n}\varphi_{y^{(n)}}$ and $\frac{d^{n-1}}{dx^{n-1}}\varphi_{y^{(n-1)}}$. We will, therefore, concentrate on the last two terms. Now: $\frac{d^{n-1}}{dx_{n-1}}\varphi_{y^{(n-1)}} = \frac{d^{n-2}}{dx_{n-2}}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}y' + \frac{\partial}{\partial y'}y'' + \cdots + \frac{\partial}{\partial y^{(n)}}y^{(n+1)}\right)\varphi_{y^{(n-1)}}$. We easily see that the only term containing $y^{(2n-1)}$ in this expression will be

$$y^{(2n-1)}\frac{\partial}{\partial y^{(n)}}\varphi_{y^{(n-1)}} = \varphi_{y^{(n-1)}y^{(n)}} \cdot y^{(2n-1)}$$

$$\frac{d^n}{dx^n}\varphi_{y^{(n)}} = \frac{d^{n-1}}{dx_{n-1}}\left(\frac{\partial}{\partial x} + \cdots + \frac{\partial}{\partial y^{(n)}}y^{(n+1)}\right)\varphi_{y^{(n)}}$$

The terms containing $y^{(2n-1)}$ and $y^{(2n)}$ will be

$$\frac{\partial}{\partial y^{(n-1)}}\varphi_{y^{(n)}}y^{(2n-1)} + \frac{d}{dx}\frac{\partial}{\partial y^{(n)}}\varphi_{y^{(n)}} \cdot y^{(2n-1)} + \frac{\partial}{\partial y^{(n)}}\varphi_{y^{(n)}} \cdot y^{(2n)}$$

$$= \left[\varphi_{y^{(n-1)}y^{(n)}} + \frac{d}{dx}\varphi_{y^{(n)}y^{(n)}}\right] y^{(2n-1)} + \varphi_{y^{(n)}y^{(n)}}\, y^{(2n)}$$

Combining the two we see that the Euler-Lagrange expressions can be written as follows:

$$y^{(2n)} = \frac{d}{dx}\operatorname{Log}\varphi_{y^{(n)}y^{(n)}} \cdot y^{(2n-1)} + \psi(x, y, y', \cdots, y^{(2n-2)}) \tag{5.39}$$

Thus it is linear in $y^{(2n-1)}$

Combining this equation with the recursion formula (5.11), we get the result:

The extremals connected with any plane problem of the n^{th} order ($n > 2$) that is with any integral of the type (5.37) form an ∞^{2n} system of curves, such

that an ∞^1 are determined by a given element of order $(2n-2)$. The locus of the $(2n-1)^{st}$ center of curvature of these ∞^1 curves is a straight line.

Note: The case $n = 2$ is exceptional, for it was pointed out by Professor E. Kasner (Aspects of Dynamics, The Princeton Colloquium, New York, 1913, reprinted 1934, pg. 95) that for that case for a given curvature element, the ∞^1 curves of the system have the property that the locus of the third center of curvature is a parabola with axis parallel to the fixed radius of curvature. This is due to the fact that for $n = 2$, $2n - 1 = n + 1$ and we can see that in the Euler-Lagrange equation $y^{(n+1)}$ enters to the n^{th} power.

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ON A NON-LINEAR DIFFERENTIAL EQUATION OF THE SECOND ORDER

By Norman Levinson

1. We shall consider the equation

$$\ddot{x} + f(x)\dot{x} + x = e(t) \tag{1.0}$$

where $e(t)$ is continuous and periodic with its least period denoted by L. $f(x)$ is continuous, except possibly at a finite number of points where $f(x \pm 0)$ exist. Subject to these conditions we shall prove the following two theorems.

Theorem I. *If $f(x) > 0$ except possibly at discrete points, and if*

$$\int^{\infty} f(x)\,dx = \infty, \qquad \left(\text{or if } \int_{-\infty} f(x)\,dx = \infty\right), \tag{1.1}$$

then (1.0) *has a periodic solution of period* L *and all other solutions of* (1.0) *tend to this solution as $t \to +\infty$.*

The condition (1.1) is used here in proving the existence of a periodic solution for (1.0). However even if (1.1) is not satisfied we can show Theorem II. *If $f(x) > 0$ except possibly at discrete points, then if* (1.0) *has a periodic solution, this solution is of period L and all other solutions tend to it as $t \to +\infty$.*

This result indicates that such a phenomenon as subharmonic resonance, for example, can occur in (1.0) only if $f(x) < 0$ for some range of values of x.

The results of Theorems I and II are equally true for the equation

$$\ddot{u} + F(\dot{u}) + u = E(t) \tag{1.2}$$

where $E(t)$ is periodic of period L and F' satisfies the same requirements as f. To see this we differentiate (1.2) and set $\dot{u} = x$, obtaining

$$\ddot{x} + F'(x)\dot{x} + x = \dot{E}(t). \tag{1.3}$$

2. If we define

$$F(x) = \int_0^x f(x)dx \tag{2.0}$$

and set

$$y = \dot{x} + F(x) \tag{2.1}$$

then (1.0) can be written as

$$\dot{x} = y - F(x), \qquad \dot{y} = -x + e(t). \tag{2.2}$$

Clearly the right members of (2.2) satisfy the Lipschitz condition in x and y and thus (2.2) possesses a unique solution $(x(t), y(t))$ passing through any point of (x, y, t) space. We now prove

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LEMMA 1. *If* $(x_1(t), y_1(t))$ *and* $(x_2(t), y_2(t))$ *are any two distinct solutions of* (2.2) *and if*

$$D(t) = [(x_2(t) - x_1(t))^2 - (y_2(t) - y_1(t))^2]^{\frac{1}{2}}, \tag{2.3}$$

then $D(t)$ *is a monotonically decreasing function of* t.

Differentiating (2.3) we have

$$\frac{dD}{dt} = \frac{(x_2 - x_1)(\dot{x}_2 - \dot{x}_1) + (y_2 - y_1)(\dot{y}_2 - \dot{y}_1)}{D}.$$

Using (2.2) this becomes

$$\begin{aligned} \frac{dD}{dt} &= \frac{-1}{D}[x_2(t) - x_1(t)][F(x_2(t)) - F(x_1(t))] \\ &= -\frac{1}{D}[x_2(t) - x_1(t)] \int_{x_1}^{x_2} f(x)\, dx. \end{aligned} \tag{2.4}$$

Since $f(x) \geqq 0$ it follows that

$$\frac{dD}{dt} \leqq 0. \tag{2.5}$$

In fact since $f(x) > 0$ except possibly at discrete points it follows that

$$\frac{dD}{dt} < 0 \quad \text{wherever} \quad x_2(t) \neq x_1(t). \tag{2.6}$$

Since $x_1(t)$ and $x_2(t)$ are distinct solutions, (2.6) is valid at an everywhere dense set of values of t. From (2.4), $\frac{dD}{dt}$ is a continuous function of t. Thus (2.5) and (2.6) imply

$$D(t + h) < D(t), \qquad h > 0.$$

This proves Lemma 1.

PROOF OF THEOREM II. Let $\bar{x}(t)$ be a periodic solution of (1.0). By substituting $\bar{x}(t)$ into (1.0) it follows at once that the period of $\bar{x}(t)$ must be some multiple of L. We denote its least period by L_1. Corresponding to $\bar{x}(t)$ we have the solution $(\bar{x}(t), \bar{y}(t))$ of (2.2). If (1.0) possesses a solution $x_0(t)$ distinct from $\bar{x}(t)$ and such that $x_0(t)$ does not tend to $\bar{x}(t)$ as $t \to +\infty$, then (2.2) has a solution $(x_0(t), y_0(t))$ which does not tend to $(\bar{x}(t), \bar{y}(t))$.

Let $D_0(t)$ denote the distance between $(\bar{x}, \bar{y})$ and (x_0, y_0). Since $D_0(t)$ is monotonically decreasing by Lemma 1, and since (x_0, y_0) does not approach $(\bar{x}, \bar{y})$ as a limit, clearly

$$\lim_{t \to \infty} D_0(t) = A > 0. \tag{2.5}$$

Let $D_n(t)$ denote the distance between $(x_0(t + nL_1), y_0(t + nL_1))$ and $(\bar{x}(t), \bar{y}(t))$. Because L_1 is a period of $(\bar{x}, \bar{y})$ and of $e(t)$ clearly,

$$D_n(t) = D_0(t + nL_1).$$

Thus by (2.5)

$$\lim_{n\to\infty} D_n(t) = A. \tag{2.6}$$

Let P_n denote the point $(x_0(nL_1), y_0(nL_1))$ and $\bar{P}$ denote $(\bar{x}(0), \bar{y}(0))$ in the (x, y) plane. From (2.6), $P_n\bar{P}$ tends to A as a limit. Thus the points $\{P_n\}$, $n \geqq 0$, must have at least one limit point. Let Q be a limit point. Let $(x(t), y(t))$ be the solution of (2.2) which passes through Q when $t = 0$. From continuity considerations it follows that (2.6) implies

$$D(t) = [(x(t) - \bar{x}(t))^2 + (y(t) - \bar{y}(t))^2]^{\frac{1}{2}} = A.$$

Thus $D(t)$ is not a decreasing function which by Lemma 1 is impossible. Therefore all solutions of (2.2) must tend to $(\bar{x}(t), \bar{y}t)$ as $t \to +\infty$.

To complete the proof of Theorem II we have only to show that $\bar{x}(t)$ has L as its period. If $L_1 \neq L$, then clearly $(\bar{x}(t - L), \bar{y}(t - L)$ is a solution of (2.2) which does not tend to $(\bar{x}(t), \bar{y}(t))$ as $t \to +\infty$. But we have just shown this to be impossible. Thus $\bar{x}(t)$ has L as its period.

3. Next we prove

LEMMA 2.[1] *The system* (2.2) *possesses at least one solution of period* L *under the conditions of Theorem I.*

We shall prove this lemma by proving the existence of a simple closed curve, C, in the (x, y) plane such that all solutions of (2.2) which intersect C cut across C from the domain exterior to C into the domain interior to C as t increases. Let P_0 be a point inside or on C. Let $(x_0(t), y_0(t))$ be the solution of (2.2) such that $(x_0(0), y_0(0))$ is the point P_0. Clearly $(x_0(t), y_0(t))$ lies inside of C for $t > 0$ In particular $(x_0(L), y_0(L))$ lies inside of C. We denote $(x_0(L), y_0(L))$ by P_1 and define a transformation T by $TP_0 = P_1$. Clearly T is continuous and transforms the domain bounded by C into its interior. Thus by the Brouwer fixed-point theorem, there exists at least one point, P, such that $TP = P$. This is equivalent to the statement that the solution of (2.2), $(x(t), y(t))$ which at $t = 0$ is at P, has period L. Thus Lemma 2 will be proved if we prove the existence of C.

We shall now prove the existence of C. In what follows we denote max $|e(t)|$ by E. C is the curve $P_0P_1P_2 \cdots P_7P_0$ in Fig. 1. We begin by locating P_0. The point P_0 with coordinates (x_0, y_0) lies on the curve $y = \dfrac{xF(x)}{2E}$, and is sufficiently far out on this curve so that $x_0 > 32E$ and

$$F(x_0) > 160\,E. \tag{3.0}$$

Condition (3.0) can be satisfied by virtue of (1.1). P_0P_1 is the solution of

$$\frac{dy}{dx} = -\frac{x}{y - \frac{1}{4}F(x)} \tag{3.1}$$

[1] Compare the proof of this lemma with the proof of Theorem II, N. Levinson, On the Existence of Periodic Solutions For Second Order Differential Equations With A Forcing Term, Journal of Math. and Physics, p. 41, Vol. XXII, June 1943.

which starts at P_0 and terminates on the curve $y = \frac{1}{2} F(x)$ in the point we denote by $P_1(x_1, y_1)$. Clearly so long as $y > \frac{1}{2} F(x)$, it follows from (3.1) that $\frac{dy}{dx} < 0$ and thus P_0P_1 does have negative slope as indicated.

From (3.1) we have

$$d(x^2 + y^2) = \tfrac{1}{2} F(x)dy \tag{3.2}$$

for P_0P_1. From (2.2) we have

$$d(x^2 + y^2) = \frac{2xF(x) - 2ye(t)}{x - e(t)}\, dy \tag{3.3}$$

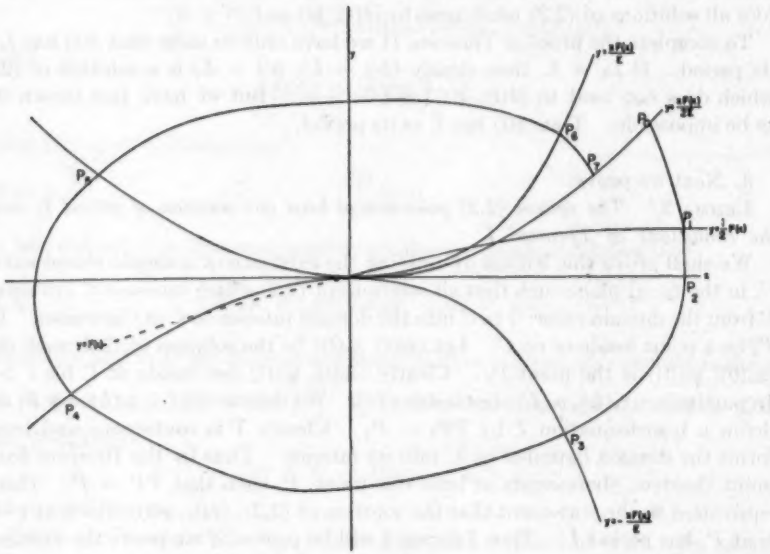

FIG. 1

for solutions of (2.2). For $dt > 0$, we have $dy < 0$ for solutions of (2.2) when $x > E$ as is the case near P_0P_1. Thus we have from (3.3)

$$d(x^2 + y^2) < \frac{2xF(x) - 2yE}{x + E}\, dy.$$

Since $2yE \leq xF(x)$ along P_0P_1 and $x > E$ we have

$$d(x^2 + y^2) < \frac{xF(x)}{2x}\, dy = \tfrac{1}{2}F(x)\, dy \tag{3.4}$$

for solutions of (2.2) where they intersect P_0P_1. Comparing (3.4) with (3.2) it follows that the solutions of (2.2) cut into the circles $x^2 + y^2 = \text{const.}$ from exterior to interior closer to the normal direction than does P_0P_1. Thus they cut P_0P_1 from the exterior to the interior of C.

P_1P_2 is the vertical line extending from P_1 to the x axis. Since by (2.2), $\dot{x} = y - F(x)$ and since $y \leqq \frac{1}{2}F(x)$ near P_1P_2, it follows that $dx < 0$ for solutions of (2.2) intersecting P_1P_2. Thus P_1P_2 is cut from the exterior of C to the interior by solutions of (2.2).

P_2P_3 is the arc of the circle with center at (0, 0) and extending from P_2 to the point P_3 where it first intersects the curve $y = \dfrac{-xF(x)}{E}$. From (2.2) we have

$$\frac{d}{dt}(x^2 + y^2) = -2xF(x) + 2ye(t). \tag{3.5}$$

From this

$$\frac{d}{dt}(x^2 + y^2) \leqq -2xF(x) + 2E|y|. \tag{3.6}$$

But along P_2P_3, $E|y| < xF(x)$ since P_2P_3 lies above the curve $y = -\dfrac{xF(x)}{E}$. Thus (3.6) becomes

$$\frac{d}{dt}(x^2 + y^2) < 0.$$

In other words the solutions of (2.2) cut P_2P_3 which is an arc of a circle $x^2 + y^2 =$ const., from the exterior of C to the interior.

P_3P_4 is the solution of

$$\frac{dy}{dx} = -\frac{4E + x}{y} \tag{3.7}$$

which starts at P_3 and terminates on the curve $y = -\dfrac{xF(x)}{E}$ in the third quadrant at a point we denote by P_4. P_4, we shall assume, is below the point where $y = -\dfrac{xF(x)}{E}$ is cut by $y = F(x)$. If P_0 is taken sufficiently remote from the origin this will certainly be the case. Thus along P_3P_4 we can assume $y < -|F(x)|$. (3.7) is equivalent to

$$\frac{d}{dx}(x^2 + y^2) = -8E. \tag{3.8}$$

For solutions of (2.2) we have from (3.5)

$$\frac{d}{dt}(x^2 + y^2) \leqq 2|xF(x)| + 2E|y|. \tag{3.9}$$

Since on P_3P_4 we have

$$|y| \geqq \frac{|xF(x)|}{E},$$

(3.9) becomes

$$\frac{d}{dt}(x^2 + y^2) \leqq 4E|y|. \tag{3.10}$$

By (2.2), $dx = (y - F(x))dt$. Since $y < F(x)$ along P_3P_4, we have $dx < 0$ for

solutions of (2.2) where they intersect P_3P_4. Using the above equation for dx, (3.10) becomes

$$\frac{d}{dx}(x^2+y^2) > \frac{-4E|y|}{|y|-|F(x)|}. \tag{3.11}$$

For $|x| > 2E$ it follows, since $|y| > \frac{|xF(x)|}{E}$ along P_3P_4 that $|F(x)| < \frac{1}{2}|y|$. Thus (3.11) becomes for $|x| > 2E$

$$\frac{d}{dx}(x^2+y^2) > -8E. \tag{3.12}$$

Denote P_3 by (x_3, y_3). For $|x| \leqq 2E$ we have $y < y_3$ along P_3P_4 by (3.7). Using $y_3 = -\frac{x_3F(x_3)}{E}$, we have therefore along P_3P_4

$$|y| > \frac{x_3F(x_3)}{E}, \qquad |x| \leqq 2E. \tag{3.13}$$

But with x_3 sufficiently large, as it will be if x_0 is taken large enough, clearly

$$\frac{x_3F(x_3)}{E} > 2|F(x)|, \qquad |x| \leqq 2E.$$

With (3.13) this gives us $|y| > 2|F(x)|$ along P_3P_4 when $|x| \leqq 2E$. Using this, (3.11) yields (3.12) for $|x| \leqq 2E$. That is, (3.12) is valid for solutions of (2.2) along the entire length of P_3P_4. Comparing (3.12) with (3.8) we see that solutions of (2.2) cut across P_3P_4 from the exterior of C to the interior.

For P_4P_5 we take the arc of the circle $x^2+y^2 = \text{const.}$ P_5 is taken on the curve $y = \frac{xF(x)}{E}$ in the second quadrant. Exactly as with P_2P_3, solutions of (2.2) cut P_4P_5 from the exterior of C to the interior.

P_5P_6 is defined as the solution of

$$\frac{dy}{dx} = \frac{4E-x}{y}$$

starting at P_5 and terminating in the curve $y = \frac{xF(x)}{E}$ in the first quadrant. Exactly as with P_3P_4 solutions of (2.2) cut P_5P_6 from the exterior of C to the interior.

P_6P_7 is the arc of the circle $x^2+y^2 = \text{const.}$ starting at P_6 and terminating on the curve $y = \frac{xF(x)}{2E}$. P_6P_7 is treated as P_2P_3 and P_4P_5.

If P_7 lies below P_0 on the curve $y = \frac{xF(x)}{2E}$, then P_7P_0 is cut by solutions of (2.2) from the exterior of C to the interior, since from (2.2), $dy < 0$, $dx > 0$, for solutions of (2.2) as they intersect P_7P_0. Thus to complete our proof of Lemma 2 we have only to show that P_7 lies below P_0.

We denote P_n by (x_n, y_n) and $x_n^2+y_n^2$ by r_n^2. From (3.2) we have on integrating,

$$r_1^2 - r_0^2 = \frac{1}{2}\int_{P_0}^{P_1} F(x)\,dy.$$

Since $dy < 0$ and $F(x)$ increases with x, we have

$$r_1^2 - r_0^2 < -\tfrac{1}{2} F(x_0)(y_0 - y_1). \tag{3.14}$$

For P_1P_2 we have

$$r_2^2 - r_1^2 = -y_1^2 = -y_1 \tfrac{1}{2} F(x_1) < -\tfrac{1}{2} y_1 F(x_0).$$

Adding to (3.14) we have

$$r_2^2 - r_0^2 < -\tfrac{1}{2} y_0 F(x_0). \tag{3.15}$$

Using (3.8) we have

$$r_4^2 - r_3^2 = -8E(x_4 - x_3) = 8E(|x_4| + |x_3|).$$

Thus

$$r_4^2 - r_3^2 < 8E(r_4 + r_3)$$

or

$$r_4 - r_3 < 8E. \tag{3.16}$$

In exactly the same way

$$r_6 - r_5 < 8E. \tag{3.17}$$

Since $r_2 = r_3$, $r_4 = r_5$, and $r_6 = r_7$, we have from (3.16) and (3.17), $r_7 - r_2 < 16E$ or

$$r_7 < r_2 + 16E.$$

Squaring the above we find

$$r_7^2 - r_2^2 < 32Er_2 + 256E^2. \tag{3.18}$$

Since $r_2 > x_0 > 32E$ we have $8Er_2 > 256E^2$ and thus (3.18) becomes

$$r_7^2 - r_2^2 < 40Er_2.$$

Adding this to (3.15) we have

$$r_7^2 - r_0^2 < -\tfrac{1}{2} y_0 F(x_0) + 40Er_2. \tag{3.19}$$

But

$$r_0^2 = y_0^2 + x_0^2 = y_0^2 + \frac{4E^2 y_0^2}{F^2(x_0)} = y_0^2\left(1 + \frac{4E^2}{F^2(x_0)}\right).$$

Thus by (3.0), $r_0 < 2y_0$. Since $r_2 < r_0$, we have $r_2 < 2y_0$. Using this in (3.19) we get

$$r_7^2 - r_0^2 < -\tfrac{1}{2} y_0[F(x_0) - 160E].$$

By (3.0) we conclude that $r_7 < r_0$ and thus P_7 lies below P_0 completing the proof of the lemma.

PROOF OF THEOREM I. Theorem I is an immediate consequence of Theorem II and Lemma 2.

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ON THE STRESS DISTRIBUTION IN CANTILEVER BEAMS

By F. B. Hildebrand

1. **Introduction.** The problem of determining the state of stress in a flanged or unflanged cantilever beam with narrow rectangular cross section, under a prescribed transverse loading in the plane of the web plate, has been dealt with as a problem of plane stress in a number of papers. The solutions presented are however approximate in most cases since they fail to satisfy exactly the condition of no warping of the built-in end as well as the physically prescribed conditions at the free end.

An exact solution was obtained in reference 1 for the case of a web plate of material entirely non-resistant to stress in the transverse direction when a transverse concentrated load is acting through a free-end stiffener, rigid in the direction of its own axis and non-resistant to bending in the plane of the web. While the exact solution of the problem for an isotropic web plate has not been given, several approximate solutions have been presented. In particular, such a solution was obtained in reference 2 by an application of the principle of least work for the case of a beam subject to a transverse concentrated end load or to a uniformly distributed load.

The present paper gives the exact solution of the problem for a web plate perfectly rigid in the direction of the applied loading. The results obtained are compared with the results of an approximate least-work solution of the same problem, found by the methods of reference 2. It is thought that a comparison of these solutions with the earlier solutions may afford information concerning the accuracy of the method of least work as applied in reference 2 and elsewhere, and may also give a further indication as to the influence of the normal stress in the transverse direction (neglected in the elementary theory) on the stress pattern in an actual beam.

2. **Formulation of the problem.** A cantilever beam with narrow rectangular cross section, with or without flanges (Fig. 1), is assumed to be clamped at one end ($x = l$) in such a way that the cross section at that end remains plane. A prescribed distribution of transverse loads is applied in the plane of the web plate in such a way that the normal stress σ_x in the longitudinal direction is antisymmetrical with respect to the center line ($y = 0$) of the beam. It is assumed that the material of the beam is rigid in the transverse (y) direction. (Such a situation would be nearly realized in the case of an isotropic beam reinforced by closely spaced rigid transverse stiffeners.)

The basic equations of plane stress to be satisfied by the stresses, strains and displacements are, with a conventional notation, the *equations of equilibrium*

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau}{\partial y} = 0, \qquad \frac{\partial \tau}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0 \tag{1a, b}$$

and the *stress-strain relations*

$$\epsilon_x = \frac{\partial u}{\partial x} = \frac{1}{E_x}(\sigma_x - \nu_x\sigma_y), \qquad \epsilon_y = \frac{\partial v}{\partial y} = \frac{1}{E_y}(\sigma_y - \nu_y\sigma_x), \tag{2a, b, c}$$

$$\gamma = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{1}{G}\tau.$$

For a material rigid in the y direction there follows

$$E_y = \infty, \qquad \nu_x = 0 \tag{3a, b}$$

and (2b) becomes

$$\frac{\partial v}{\partial y} = 0, \qquad v = v(x). \tag{4}$$

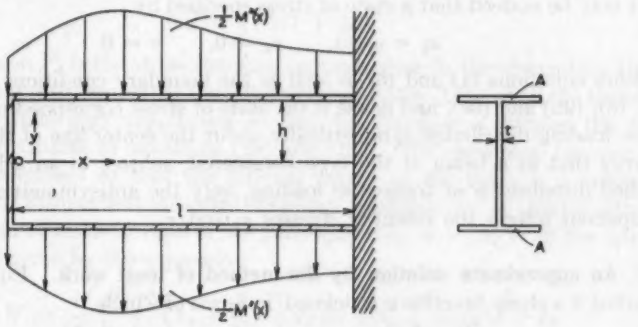


FIG. 1. Sketch of Flanged Cantilever Beam

If E is written for E_x, equations (2a) and (2c) take the form

$$E\frac{\partial u}{\partial x} = \sigma_x, \qquad G\frac{\partial u}{\partial y} = \tau - Gv'(x) \tag{5a, b}$$

from which there follows, by differentiation, the *equation of compatibility*

$$\frac{\partial \tau}{\partial x} - \frac{G}{E}\frac{\partial \sigma_x}{\partial y} = Gv''(x) \tag{6}$$

or

$$\frac{\partial}{\partial y}\left(\frac{\partial \tau}{\partial x} - \frac{G}{E}\frac{\partial \sigma_x}{\partial y}\right) = 0. \tag{6a}$$

The boundary conditions of the problem are (Fig. 1)

$$x = 0: \qquad \sigma = 0, \qquad v = \text{const.} \tag{7a, b}$$

$$x = l: \qquad \frac{\partial u}{\partial y} = \text{const.}, \qquad v = \text{const.} \tag{8a, b}$$

$$y = \pm h: \quad t\sigma_y = \mp\tfrac{1}{2}M''(x), \qquad A\frac{\partial}{\partial x}\sigma_{\text{flange}} = \pm t\tau, \qquad \epsilon_{\text{flange}} = \epsilon_x. \tag{9a, b, c}$$

The function $M(x)$ represents the moment of the applied transverse loads at a section x = constant. Conditions (8a) and (8b) state that the fixed end section is to remain plane, while conditions (9b) and (9c) require stress equilibrium and strain continuity at the junctions of the beam and flanges. Since equations (2a) and (3b) give

$$\epsilon_x = \frac{1}{E}\sigma_x \tag{10}$$

it follows that continuity of the strain implies continuity of the spanwise normal stress, so that equation (9c) is equivalent to the equation

$$\sigma_{\text{flange}} = \sigma_x(x, \pm h). \tag{11}$$

Equations (7b) and (8b) are automatically satisfied in virtue of (4).

It may be noticed that a state of stress specified by

$$\sigma_y = \sigma_y(x), \qquad \sigma_x = 0, \qquad \tau = 0 \tag{12}$$

satisfies equations (1) and (6) as well as the boundary conditions of equations (7), (8), (9b) and (9c), and hence is the state of stress corresponding to a transverse loading distributed symmetrically about the center line of the beam. It follows that in a beam of the type considered, subject to an arbitrarily prescribed distribution of transverse loading, only the antisymmetrical (bending) component affects the essential stresses σ_x and τ.

3. **An approximate solution by the method of least work.** Equation (1) is satisfied if a stress function φ is defined in terms of which

$$\sigma_x = \frac{\partial^2\varphi}{\partial y^2}, \qquad \tau = -\frac{\partial^2\varphi}{\partial x\,\partial y}, \qquad \sigma_y = \frac{\partial^2\varphi}{\partial x^2}. \tag{13a, b, c}$$

Equation (6) then requires that φ satisfy the differential equation

$$\frac{\partial^2}{\partial y^2}\left\{\frac{\partial^2\varphi}{\partial x^2} + \frac{G}{E}\frac{\partial^2\varphi}{\partial y^2}\right\} = 0. \tag{14}$$

The boundary conditions (7a), (8a) and (9) can now be expressed in the form

$$\varphi(0, y) = 0, \qquad 2th\varphi_{xy}(l, y) = -M'(l), \tag{15a, b}$$

$$\varphi(x, 0) = 0, \qquad \alpha h\varphi_{yy}(x, h) + \varphi_y(x, h) = 0, \tag{16a, b}$$

$$2t\varphi(x, h) = -M(x). \tag{17}$$

In these equations an irrelevant additive linear function of x and y has been disregarded in the expression for φ. The parameter α is the ratio of the total flange area $2A$ to the beam cross-sectional area $2th$,

$$\alpha = \frac{A}{th}. \tag{18}$$

In accordance with the procedure of reference 2, an approximation to φ is assumed in the form

$$\varphi = F_*(x, y) + F(x, y) = f_*(x)g_*(y) + f(x)g(y) \tag{19}$$

where

$$f_e(x) = \frac{M(x)h}{I}, \qquad g_e(y) = h^2\left[\frac{1}{6}\left(\frac{y}{h}\right)^3 - \left(\alpha + \frac{1}{2}\right)\frac{y}{h}\right], \tag{20a, b}$$

$$g(y) = h^2\left[\frac{1+7\alpha}{1+3\alpha}\frac{y}{h} - 2\frac{1+5\alpha}{1+3\alpha}\left(\frac{y}{h}\right)^3 + \left(\frac{y}{h}\right)^5\right], \tag{20c}$$

and the function $f(x)$ satisfies the end condition

$$f(0) = 0 \tag{21}$$

but is otherwise unspecified.

The factor I is the principal moment of inertia of the cross section,

$$I = \tfrac{2}{3}\, h^3 t(1 + 3\alpha). \tag{22}$$

The expression F_e is the stress function corresponding to the elementary theory and in general satisfies neither the differential equation (14) nor all the boundary conditions, while $g(y)$ was chosen so that the function φ satisfies the stress boundary conditions of equations (16) and (17). The remaining function $f(x)$ is to be determined so that the internal energy of the beam is a minimum. Equation (21) insures that the stress boundary condition (7a) or (15a) is satisfied.

For a beam of material rigid in the y direction ($E_y = \infty$, $\nu_z = 0$) the internal energy W is given by the expression

$$W = \frac{t}{2E}\int_0^l \int_{-h}^{h}\left(\sigma_x^2 + \frac{E}{G}\tau^2\right) dy\, dx + \frac{A}{E}\int_0^l [\sigma_x(x, h)]^2\, dx \tag{23}$$

where the double integral represents the energy stored in the web plate and the single integral gives the energy stored in the two flanges. A necessary condition that W be a minimum is that its variation vanish,

$$\delta W = 0. \tag{24}$$

If the expressions of equation (13) are introduced into (23) and the result is transformed by repeated integration by parts, equation (24) can be written in the form

$$\int_0^l \int_{-h}^{h} \frac{\partial^2}{\partial y^2}\left[\varphi_{xx} + \frac{G}{E}\varphi_{yy}\right]\delta\varphi\, dy\, dx - \int_0^l \left\{\left[\varphi_{xxy} + \frac{G}{E}\varphi_{yyy}\right]\delta\varphi\right\}_{-h}^{h} dx$$
$$+ \frac{G}{E}\int_0^l \left\{[\varphi_{yy}\,\delta\varphi_y]_{-h}^{h} + \frac{2A}{t}\sigma_x(x, h)\delta\sigma_x(x, h)\right\} dx - \int_{-h}^{h} \{\varphi_{xyy}\,\delta\varphi\}_0^l\, dy$$
$$+ \{[\varphi_{xy}\,\delta\varphi]_{-h}^{h}\}_0^l = 0. \tag{25}$$

Since the varied stresses must satisfy the same boundary conditions as the stresses themselves, there follows from (15a), (16b) and (17)

$$\delta\varphi(0, y) = 0, \qquad \left[\pm\frac{A}{t}\,\delta\varphi_x + \delta\varphi_y\right]_{y=\pm h} = 0, \qquad \delta\varphi(x, \pm h) = 0. \tag{26}$$

Thus all the single integrals in (25) vanish identically with the exception of the one taken along $x = l$, and (25) becomes

$$\int_0^l \int_{-h}^{h} \frac{\partial^2}{\partial y^2}\left[\varphi_{xx} + \frac{G}{E}\varphi_{yy}\right] \delta\varphi \, dy \, dx - \int_{-h}^{h} \varphi_{xyy}(l, y)\delta\varphi(l, y) \, dy = 0 \qquad (27)$$

or

$$\int_0^l \left\{ \int_{-h}^{h}\left[f_e'' g_e'' + f'' g'' + \frac{G}{E} f g^{IV}\right] g \, dy\right\}\delta f \, dx$$

$$- \left\{ \int_{-h}^{h} [f_e'(l) g_e'' + f'(l) g''] g \, dy\right\} \delta f(l) = 0. \qquad (27a)$$

The integrations with respect to y can be carried out, after which, because of the arbitrariness of the variations $\delta f(x)$ and $\delta f(l)$, the coefficients of these variations must vanish independently. The results of this process, in addition to (21), are the equations

$$f''(x) - \frac{\mu^2}{l^2} f(x) = \frac{\lambda^2}{120} f_e''(x), \qquad (28)$$

$$f(0) = 0, \qquad f'(l) = \frac{\lambda^2}{120} f_e'(l). \qquad (29a, b)$$

where

$$\lambda^2 = 45 \frac{(1 + 3\alpha)(1 + 10\alpha)}{2 + 30\alpha + 135\alpha^2}, \qquad \mu^2 = \frac{G}{E}\left(\frac{l}{h}\right)^2 \lambda^2. \qquad (30a, b)$$

The differential equation (28) takes the place of equation (14) in an exact analysis, while (29b) replaces (15b).

The solution of equations (28) and (29) is found in the form

$$f(x) = \frac{\lambda^2}{120\mu}\left\{\frac{\sinh \mu\xi}{\cosh \mu}\left[l f_e'(l) - \int_0^1 \cosh \mu(1 - \eta) \frac{d^2 f_e}{d\eta^2} \, d\eta\right]\right.$$

$$\left. + \int_0^\xi \sinh \mu(\xi - \eta) \frac{d^2 f_e}{d\eta^2} \, d\eta\right\}, \qquad \xi = \frac{x}{l}. \qquad (31)$$

In particular, for the case of a *concentrated transverse end load* P there follows

$$M(x) = \sigma_0 \frac{I}{h} \frac{x}{l}, \qquad f_e(x) = \sigma_0 \frac{x}{l} \qquad (32)$$

where σ_0 is the maximum normal stress according to the elementary theory,

$$\sigma_0 = \frac{Plh}{I}. \qquad (33)$$

Equation (31) then gives

$$\frac{f(x)}{\sigma_0} = \frac{\lambda^2}{120\mu} \frac{\sinh \mu \frac{x}{l}}{\cosh \mu} = \sqrt{\frac{E}{G}} \frac{h}{l} \frac{\lambda}{120} \frac{\sinh \mu \frac{x}{l}}{\cosh \mu}. \qquad (34)$$

The approximate stresses in the beam are thus derived in this case, according to equations (13a, b, c), from the function

$$\varphi = h^2\sigma_0\left[\frac{1}{2}\frac{x}{l}\left\{\frac{1}{3}\left(\frac{y}{h}\right)^3 - (1+2\alpha)\frac{y}{h}\right\}\right.$$

$$\left. + \sqrt{\frac{E}{G}}\frac{h}{l}\frac{\lambda}{120}\frac{\sinh\mu\,\frac{x}{l}}{\cosh\mu}\left\{\frac{1+7\alpha}{1+3\alpha}\frac{y}{h} - 2\frac{1+5\alpha}{1+3\alpha}\left(\frac{y}{h}\right)^3 + \left(\frac{y}{h}\right)^5\right\}\right]. \quad (35)$$

The ratio of the maximum stress $\sigma_x(l, h)$ to the maximum elementary stress σ_0 is of particular interest, and is given by

$$\frac{\sigma_x(l,h)}{\sigma_0} = 1 + \frac{\lambda}{15(1+3\alpha)}\sqrt{\frac{E}{G}}\frac{h}{l}\tanh\mu. \quad (36)$$

4. **The exact solution.** It will be convenient here to satisfy (1a) by expressing σ_x and τ in terms of a stress function H such that

$$\sigma_x = \frac{\partial H}{\partial y}, \qquad \tau = -\frac{\partial H}{\partial x}. \quad (37a, b)$$

This function is related to the function φ of equation (12) by the equation

$$H = \frac{\partial \varphi}{\partial y}.$$

The equation of compatibility, (6), then becomes

$$\frac{\partial}{\partial y}\left\{\frac{\partial^2 H}{\partial x^2} + \frac{G}{E}\frac{\partial^2 H}{\partial y^2}\right\} = 0. \quad (38)$$

If H is determined, the stress σ_y can be found from equation (1b),

$$\sigma_y = \int_0^y \frac{\partial^2 H}{\partial x^2}\,dy + f_1(x) = -\frac{G}{E}\frac{\partial H}{\partial y} - Gv''(x) + f_2(x) \quad (39)$$

where f_1 and f_2 are functions of integration which vanish when the loading is antisymmetrical as in figure 1. The boundary conditions take the form

$$H(0, y) = 0, \qquad 2thH_x(l, y) = -M'(l), \quad (40a, b)$$

$$H_y(x, 0) = 0, \qquad \alpha hH_y(x, h) + H(x, h) = 0, \quad (41a, b)$$

$$2t\int_0^h H(x, y)\,dy = -M(x), \quad (42)$$

if an irrelevant additive constant is disregarded in the expression for H.

The solution of the problem is accomplished by first writing

$$H(x, y) = F_e(x, y) + F(x, y) \quad (43)$$

where the function F_e is a conveniently chosen function satisfying the differential equation (38) and the boundary conditions (41) and (42), and by then deter-

mining the function F so that H satisfies (38), (40), (41) and (42). Since (41) and (42) are now homogeneous in F, this procedure leads to a problem of characteristic values.

a. *An expression for* F_s. It can be directly verified that the expression

$$F_s(x, y) = f_s(x)g_2(y) - h^2\frac{E}{G}f_s''(x)[g_4(y) - c_0 g_2(y)]$$

$$+ h^4\left(\frac{E}{G}\right)^2 f_s^{IV}(x)[g_6(y) - c_0 g_4(y) + c_1 g_2(y)]$$

$$- h^6\left(\frac{E}{G}\right)^3 f_s^{VI}(x)[g_8(y) - c_0 g_6(y) + c_1 g_4(y) - c_2 g_2(y)] + \cdots \quad (44)$$

where

$$f_s(x) = \frac{M(x)h}{I}, \qquad g_n(y) = \frac{h}{n!}\left[\left(\frac{y}{h}\right)^n - (1 + n\alpha)\right] \quad (45a, b)$$

formally satisfies (38) and (41) for arbitrary values of the constants c_n. Furthermore, the first term of (44) satisfies (42). Hence it remains only to determine successively the values of the constants c_n so that the coefficients of f_s'', f_s^{IV}, $\cdots$, vanish independently when the expression of equation (44) is introduced into the left-hand side of (42). The first two constants are determined in this way by the equations

$$\int_0^h [g_4 - c_0 g_2]\,dy = 0, \qquad c_0 = \frac{1}{10}\frac{1 + 5\alpha}{1 + 3\alpha}; \quad (46a)$$

$$\int_0^h [g_6 - c_0 g_4 + c_1 g_2]\,dy = 0, \qquad c_1 = \frac{1}{1400}\frac{9 + 90\alpha + 245\alpha^2}{(1 + 3\alpha)^2}; \quad (46b)$$

and the following constants are similarly determined.

Thus when $M(x)$ is a polynomial a function F_s satisfying (38), (41) and (42) is given by (44) as a finite polynomial in x and y. The convergence of (44) for more general classes of functions M will not be considered in this paper.

b. *The determination of* F. In view of the fact that F_s satisfies (38), (41) and (42), the boundary value problem in F becomes

$$\frac{\partial}{\partial y}\left\{\frac{\partial^2 F}{\partial x^2} + \frac{G}{E}\frac{\partial^2 F}{\partial y^2}\right\} = 0, \quad (47)$$

$$F(0, y) = -F_s(0, y), \qquad 2thF_x(l, h) = -[M'(l) + 2thF_{s,x}(l, y)], \quad (48a, b)$$

$$F_y(x, 0) = 0, \qquad \alpha hF_y(x, h) + F(x, h) = 0, \quad (49a, b)$$

$$\int_0^h F(x, y) = 0. \quad (50)$$

If a formal expansion of the form

$$F(x, y) = \sum_n X_n(x)Y_n(y) \quad (51)$$

is assumed, the differential equation (47) can be separated into the two ordinary differential equations

$$X_n'' - \frac{\mu_n^2}{l^2} X_n = 0, \tag{52}$$

$$Y_n''' + \frac{\lambda_n^2}{h^2} Y_n' = 0, \tag{53}$$

where μ_n and λ_n are arbitrary constants satisfying the relationship

$$\mu_n = \sqrt{\frac{G}{E}} \frac{l}{h} \lambda_n . \tag{54}$$

The boundary conditions become

$$Y_n'(0) = 0, \qquad \alpha h Y_n'(h) + Y_n(h) = 0, \qquad \int_0^h Y_n \, dy = 0 \tag{55a, b, c}$$

and

$$\left.\begin{aligned} \sum_n X_n(0) Y_n(y) &= -F_s(0, y), \\ \sum_n X_n'(l) Y_n(y) &= -\left[\frac{1}{2th} M'(l) + F_{s,x}(l, y)\right]. \end{aligned}\right\} \tag{56a, b}$$

From equations (53) and (55) there follows

$$Y_n(y) = C_n\left(\cos \lambda_n \frac{y}{h} - \frac{\sin \lambda_n}{\lambda_n}\right) \tag{57}$$

where the constants C_n are arbitrary and the parameters λ_n are the solutions of the transcendental equation

$$\tan \lambda_n = \frac{\lambda_n}{1 + \alpha \lambda_n^2} . \tag{58}$$

(A table of values of the first ten solutions of this equation, for several values of α, is presented in table 1.) It is convenient to choose the constants C_n in such a way that the functions Y_n are normalized over $0 \leq y \leq h$,

$$\int_0^h Y_n^2 \, dy = 1, \tag{59}$$

so that

$$C_n = \left[\int_0^h \left\{\cos \lambda_n \frac{y}{h} - \frac{\sin \lambda_n}{\lambda_n}\right\}^2 dy\right]^{-1} = \frac{2}{h}\left[1 + \alpha \sin^2 \lambda_n - \left(\frac{\sin \lambda_n}{\lambda_n}\right)^2\right]^{-1}. \tag{60}$$

The general solution of (52) is

$$X_n = A_n \sinh \mu_n \frac{x}{l} + B_n \cosh \mu_n \frac{x}{l} \tag{61}$$

where A_n and B_n are constants of integration, so that equations (56a) and (56b) become

$$\sum_n B_n Y_n(y) = -F_*(0, y), \tag{62a}$$

$$\sum_n \mu_n(A_n \cosh \mu_n + B_n \sinh \mu_n)Y_n(y) = -l\left[\frac{1}{2lh} M'(l) + F_{*,x}(l, y)\right] \tag{62b}$$

for the interval $0 \leq y \leq h$.

It can be shown by conventional methods that the functions $Y_n(y)$ are orthogonal in the interval $0 \leq y \leq h$ and, further, that any function $f(y)$ satisfying the condition

$$\int_0^h f(y)\, dy = 0, \tag{63}$$

TABLE 1

Solutions of the equation $\tan \lambda_n = \dfrac{\lambda_n}{1 + \alpha\lambda_n^2}$

n	λ_n				
	$\alpha = 0$	$\alpha = 0.1$	$\alpha = 1$	$\alpha = 5$	$\alpha = \infty$
0	0	0	0	0	0
1	4.4934	4.1324	3.4056	3.2028	3.1416
2	7.7252	7.1469	6.4338	6.3147	6.2832
3	10.9041	10.1563	9.5272	9.4459	9.4248
4	14.0662	13.1884	12.6448	12.5822	12.5664
5	17.2208	16.2433	15.7710	15.7207	15.7080
6	20.3713	19.3166	18.9023	18.8602	18.8496
7	23.5194	22.4037	22.0364	22.0002	21.9911
8	26.6661	25.5013	25.1724	25.1407	25.1327
9	29.8116	28.6068	28.3096	28.2814	28.2743
10	32.9564	31.7184	31.4477	31.4223	31.4159

and any of the conditions sufficient for expansibility in a Fourier series, is representable in a convergent series of the form

$$\sum_{n=1}^{\infty} a_n Y_n(y) = f(y), \qquad 0 < y < h, \tag{64}$$

where, if (59) is satisfied, the coefficients a_n are given by

$$a_n = \int_0^h f(y)Y_n(y)\, dy. \tag{65}$$

Since equations (46a, b, $\cdots$) require that

$$\int_0^h [F_*(x, y) - f_*(x)g_2(y)]\, dy = 0 \tag{66}$$

and since $f_s(0) = 0$ for a beam unsupported at the end $x = 0$, it follows that the right-hand side of (62a) satisfies (63). Similarly, from (66), (45) and (22) there follows

$$\int_0^h \left[\frac{1}{2th} M'(l) + F_{s,x}(l, y)\right] dy = \int_0^h \left[\frac{1}{2th} M'(l) + f_s'(l) g_2(y)\right] dy = \frac{h}{2} f_s'(l) \int_0^h \left[\left(\frac{y}{h}\right)^2 - \frac{1}{3}\right] dy = 0 \tag{67}$$

TABLE 2

Values of the stress-concentration factor $\frac{\sigma_x(l, h)}{\sigma_0}$

$\frac{2h}{l}$	$\sigma_x(l, h)/\sigma_0$			
	$E_x = E_y$ Least work	$E_y = \infty$		$E_y = 0$ Exact
		Least work	Exact	
		$\alpha = 1.0$		
0	1.0125	1.0000	1.0000	1.0000
.2	1.0200	1.0094	1.0106	1.0013
.5	1.0312	1.0234	1.0265	1.0083
.7	1.0387	1.0328	1.0372	1.0147
1.0	1.0499	1.0469	1.0531	1.0288
		$\alpha = 0.1$		
0	1.0603	1.0000	1.0000	1.0000
.2	1.0886	1.0359	1.0519	1.0041
.5	1.1311	1.0898	1.1299	1.0250
.7	1.1593	1.1258	1.1818	1.0466
1.0	1.2018	1.1797	1.2597	1.0946

so that the right-hand side of (62b) also satisfies (63). Hence, if F_s has sufficient regularity (in particular, if $M(x)$ is a polynomial) equations (62a) and (62b) imply the conditions

$$B_n = -\int_0^h F_s(0, y) Y_n(y)\, dy \tag{68a}$$

and

$$A_n \cosh \mu_n + B_n \sinh \mu_n = -\frac{l}{\mu_n} \int_0^h \left[\frac{1}{2th} M'(l) + F_{s,x}(l, y)\right] Y_n(y)\, dy. \tag{68b}$$

The final solution of the problem is then given by the stress function

$$H(x, y) = F_s(x, y) + \sum_{n=1}^{\infty} \left(A_n \sinh \mu_n \frac{x}{l} + B_n \cosh \mu_n \frac{x}{l}\right) Y_n(y) \tag{69}$$

where the functions F_s and Y_n are defined in (44), (57) and (60) and the constants A_n and B_n are defined in (68). It should be pointed out that the stress pattern derived from the function F_s corresponds to a state of stress in a beam subjected to certain end loadings which are statically equivalent to the forces actually present and, according to the principle of Saint Venant, if the length-height ratio of the beam is sufficiently large this pattern will agree closely with the actual pattern except in regions near the ends of the beam. Hence, in such cases the series part of (69) is of the nature of a local correction and is appreciable only near the beam ends $x = 0$ and $x = l$. In particular, if $M(x)$ is an odd function of x it follows that $F_s(0, y) = 0$ and hence $B_n = 0$, so that only a fixed-end correction is represented.

For the special case of a *concentrated transverse end load* the following expressions are found:

$$M(x) = \sigma_0 \frac{I}{h}\frac{x}{l}, \qquad f_s(x) = \sigma_0 \frac{x}{l}, \qquad \sigma_0 = \frac{Plh}{I}, \tag{70}$$

$$F_s(x, y) = \frac{h\sigma_0}{2}\frac{x}{l}\left[\left(\frac{y}{h}\right)^2 - (2\alpha + 1)\right], \tag{71}$$

$$A_n = -\frac{h\sigma_0 Cn}{2\mu_n \cosh \mu_n}\int_0^h \left[\left(\frac{y}{h}\right)^2 - \frac{1}{3}\right]\left[\cos \lambda_n \frac{y}{h} - \frac{\sin \lambda_n}{\lambda_n}\right] dy$$

$$= -\frac{2}{3}h(1 + 3\alpha)\sigma_0 \frac{\sin \lambda_n}{\lambda_n \mu_n \cosh \mu_n \left\{1 + \alpha \sin \lambda_n - \left(\frac{\sin \lambda_n}{\lambda_n}\right)^2\right\}}, \tag{72}$$

$$B_n = 0. \tag{73}$$

The relevant stress function is thus obtained in the form

$$H(x, y) = h\sigma_0\left[\frac{1}{2}\frac{x}{l}\left\{\left(\frac{y}{h}\right)^2 - (1 + 2\alpha)\right\}\right.$$

$$\left. - \frac{2}{3}(1 + 3\alpha)\sum_{n=1}^{\infty} \frac{(\sin \lambda_n)\left(\cos \lambda_n \frac{y}{h} - \frac{\sin \lambda_n}{\lambda_n}\right)}{\lambda_n\left\{1 + \alpha \sin^2 \lambda_n - \left(\frac{\sin \lambda_n}{\lambda_n}\right)^2\right\}} \frac{\sinh \mu_n \frac{x}{2}}{\mu_n \cosh \mu_n}\right] \tag{74}$$

where

$$\left.\begin{aligned} \tan \lambda_n &= \frac{\lambda_n}{1 + \alpha\lambda_n^2}, \qquad \lambda_n > 0, \\ \alpha &= \frac{A}{Ih}, \qquad \mu_n = \sqrt{\frac{G}{E}}\frac{l}{h}\lambda_n. \end{aligned}\right\} \tag{75}$$

The stresses σ_x and τ follow by differentiation, in accordance with (37)

$$\sigma_x = \sigma_0 \left[\frac{x}{l}\frac{y}{h} + \frac{2}{3}(1+3\alpha)\sqrt{\frac{E}{G}}\frac{h}{l}\sum_{n=1}^{\infty} \frac{\sin\lambda_n \sin\lambda_n \frac{y}{h}}{\lambda_n\left\{1+\alpha\sin^2\lambda_n - \left(\frac{\sin\lambda_n}{\lambda_n}\right)^2\right\}} \frac{\sinh\mu_n\frac{x}{l}}{\cosh\mu_n}\right], \tag{76a}$$

$$\tau = -\frac{1}{2}\frac{h}{l}\sigma_0\left[\left(\frac{y}{h}\right)^2 - (1+2\alpha) - \frac{4}{3}(1+3\alpha)\sum_{n=1}^{\infty} \frac{\sin\lambda_n\left(\cos\lambda_n\frac{y}{h} - \frac{\sin\lambda_n}{\lambda_n}\right)}{\lambda_n\left\{1+\alpha\sin^2\lambda_n - \left(\frac{\sin\lambda_n}{\lambda_n}\right)^2\right\}} \frac{\cosh\mu_n\frac{x}{l}}{\cosh\mu_n}\right]. \tag{76b}$$

The convergence and validity of the solution can be established if account is taken of the fact that for large n, if $\alpha \neq 0$,

$$\lambda_n \sim n\pi, \qquad \cos\lambda_n \sim (-1)^n, \qquad \sin\lambda_n \sim \frac{(-1)^n}{n\pi\alpha}. \tag{77}$$

In the limiting case of an unflanged beam ($\alpha = 0$) equations (76a, b) can be written

$$\sigma_x = \sigma_0\left[\frac{x}{l}\frac{y}{h} + \frac{2}{3}\sqrt{\frac{E}{G}}\frac{h}{l}\sum_{n=1}^{\infty}\frac{\sin\lambda_n\frac{y}{h}}{\lambda_n\sin\lambda_n}\frac{\sinh\mu_n\frac{x}{l}}{\cosh\mu_n}\right], \tag{78a}$$

$$\tau = -\frac{1}{2}\frac{h}{l}\sigma_0\left[\left(\frac{y}{h}\right)^2 - 1 - \frac{4}{3}\sum_{n=1}^{\infty}\frac{\cos\lambda_n\frac{y}{h} - \cos\lambda_n}{\lambda_n\sin\lambda_n}\frac{\cosh\mu_n\frac{x}{l}}{\cosh\mu_n}\right], \tag{78b}$$

where the parameters λ_n are the positive solutions of the equation

$$\tan\lambda_n = \lambda_n. \tag{79}$$

In this case, for large values of n,

$$\lambda_n \sim n\pi + \frac{\pi}{2}, \qquad \cos\lambda_n \sim \frac{(-1)^n}{n\pi}, \qquad \sin\lambda_n \sim (-1)^n. \tag{80}$$

The series in (78a) diverges at the point (l, h) and so predicts infinite stress concentrations at the fixed corners of an unflanged beam of the type considered.

The ratio of the maximum stress $\sigma_x(l, h)$ to the maximum elementary stress σ_0 can be written

$$\frac{\sigma_x(l, h)}{\sigma_0} = 1 + \frac{2}{3}\sqrt{\frac{E}{G}}\frac{h}{l}(1 + 3\alpha)\sum_{n=1}^{\infty}\frac{\tanh \mu_n}{\lambda_n\{(1 + 3\alpha) + \alpha^2\lambda_n^2\}} \tag{81}$$

if use is made of the identity

$$\frac{\sin^2 \lambda_n}{1 + \alpha \sin^2\lambda_n - \left(\frac{\sin \lambda_n}{\lambda_n}\right)^2} = \frac{1}{(1 + 3\alpha) + \alpha^2\lambda_n^2} \tag{82}$$

which is true in virtue of (58). Since always $\lambda_n > \pi$, there follows from (54)

$$\mu_n > \pi\sqrt{\frac{G}{E}}\frac{l}{h} > 3.84\left(\frac{l}{2h}\right) \tag{83}$$

if the ratio of the elastic moduli has the value

$$\frac{G}{E} = \frac{3}{8}. \tag{84}$$

Thus for a beam with height-length ratio $(2h/l)$ less than unity the factor $\tanh \mu_n$ can be replaced by unity and equation (81) shows that for such beams the ratio of the maximum stress to the elementary maximum stress is (very nearly) a linear function of the height-length ratio.

5. **The stress distribution in a flanged cantilever beam of material non-resistant to stress in the transverse direction.** For the purpose of comparison the exact solution of the problem for a beam of material non-resistant to stresses in the y direction ($E_y = 0$, $\nu_y = 0$) is listed here. (A web plate containing axial corrugations of small pitch and depth would have approximately the property mentioned.) Boundary conditions (7), (8) and (9) are again imposed and a concentrated transverse load P is assumed to be acting at the free end of the beam. The solution is obtained by the methods of reference 1, in which only the case of an unflanged beam ($\alpha = 0$) was considered.

For such a beam the normal stress in the y direction must vanish,

$$\sigma_y = 0, \tag{85}$$

while the remaining stresses are found in the form

$$\sigma_x = \frac{\sigma_0}{3}\frac{\kappa^2(1 + 3\alpha)}{\alpha\kappa^2 \tanh \kappa + (\kappa - \tanh \kappa)}\frac{x}{l}\frac{\sinh \kappa \frac{y}{h}}{\cosh \kappa} \tag{86a}$$

$$\tau = -\kappa\frac{h}{l}\frac{\sigma_0}{6}\frac{\alpha\kappa \tanh \kappa + \left(1 - \frac{\cosh \kappa \frac{y}{h}}{\cosh \kappa}\right)}{\alpha\kappa^2 \tanh \kappa + (\kappa - \tanh \kappa)} \tag{86b}$$

where, in addition to the constants previously defined, the parameter κ is defined by the equation

$$\kappa = \sqrt{\frac{3E}{G}}\,\frac{h}{l}. \tag{87}$$

The ratio of the exact and elementary maximum stresses can be written in the form

$$\frac{\sigma_x(l, h)}{\sigma_0} = 1 + \frac{1}{3}\,\frac{\kappa^2 \tanh \kappa - 3(\kappa - \tanh \kappa)}{\alpha\kappa^2 \tanh \kappa + (\kappa - \tanh \kappa)}. \tag{88}$$

For small values of the height-length ratio this expression becomes approximately

$$\frac{\sigma_x(l, h)}{\sigma_0} \approx 1 + \frac{1}{5(1 + 3\alpha)}\,\frac{E}{G}\left(\frac{h}{l}\right)^2. \tag{89}$$

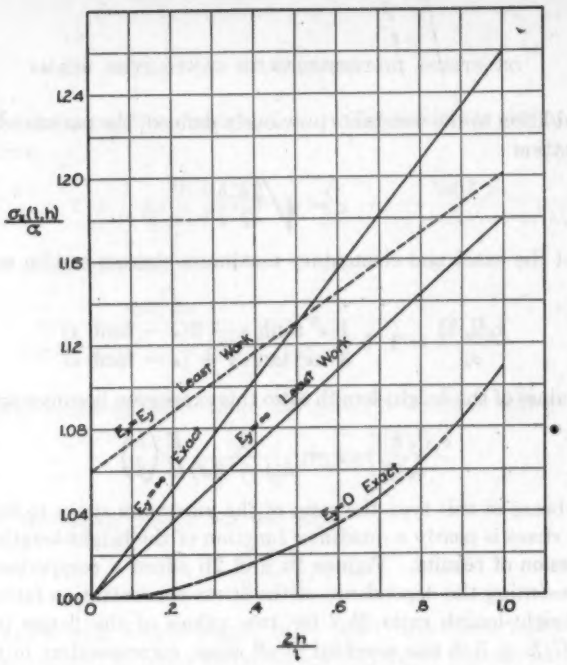
Thus for a beam of this type the ratio of the maximum stress to the maximum elementary stress is nearly a quadratic function of the height-length ratio.

6. **Discussion of results.** Figures 2a and 2b afford a comparison of several curves representing the dependence of the stress-concentration factor $\sigma_x(l, h)/\sigma_0$ upon the height-length ratio $2h/l$ for two values of the flange parameter α. The value $G/E = 3/8$ was assumed in all cases, corresponding in the isotropic case to a value of 1/3 for Poisson's ratio.

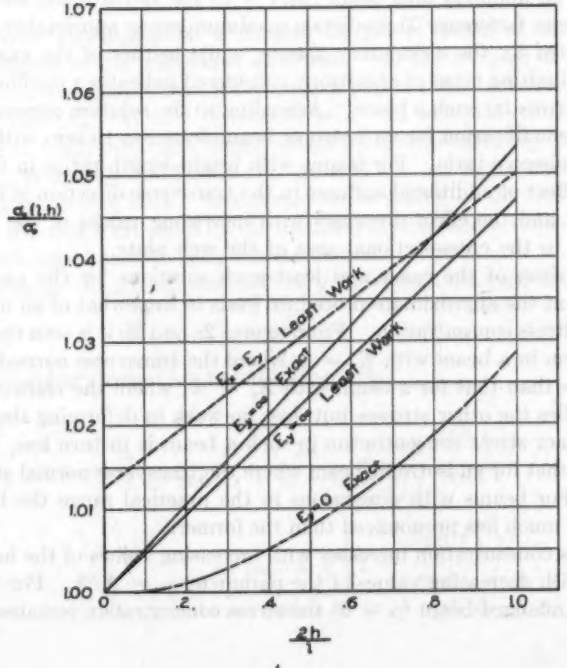
A notable difference between the curves consists in the fact that for the limiting case of an infinitely long beam ($2h/l = 0$) the approximate solution for an isotropic beam (reference 2) predicts a maximum stress appreciably higher than that predicted by the elementary theory, while neither of the exact solutions for the two limiting types of anisotropy considered indicates a modification in the maximum stress for such a beam. According to the solution presented in reference 2 this modification for an isotropic beam decreases to zero with decreasing values of Poisson's ratio. For beams with height-length ratios in the practical range the effect of additional stiffness in the transverse direction is however less pronounced and the effect decreases with increasing values of the ratio of the flange area to the cross-sectional area of the web plate.

A comparison of the exact and least-work solutions for the case $E_y = \infty$ indicates that the approximate procedure leads to somewhat of an underestimation of the stress concentration. From figures 2a and 2b it is seen that the stress concentration in a beam with $E_y = 0$, where the transverse normal stress vanishes, is less than that for a beam with $E_y = \infty$, where the transverse normal stress modifies the other stresses but does no work in deforming the beam, and that the exact stress concentration in such a beam is in turn less, in practical cases, than that for an isotropic beam where the transverse normal stress is fully effective. For beams with dimensions in the practical range the latter difference is very much less pronounced than the former.

The stress concentration increases with increasing values of the height-length ratio and with decreasing values of the parameter $\alpha = A/th$. For the limiting case of an unflanged beam ($\alpha = 0$) the stress concentration remains finite for a



a. $\frac{A}{th} = \frac{1}{10}$



b. $\frac{A}{th} = 1$

FIG. 2. Values of Ratio of Maximum Stress $\sigma_x(l, h)$ and Maximum Stress of Elementary Theory σ_0

material non-resistant to stresses in the y direction but becomes infinite for a material rigid in that direction. While the approximate solution of reference 2 predicts a finite stress concentration for an unflanged isotropic beam, it is probable, as has been mentioned in reference 2, that an exact analysis would indicate infinite stresses at the fixed corners of the beam.

A quantitative comparison of the stress in the interior of the beam, as derived from equation (74), with the corresponding pattern obtained in reference 2 for an isotropic web plate will not be included in this paper. It appears probable that for flanged beams the deviations between the least-work solution and the exact solution will be localized in small regions near the corners of the end sections, and that the magnitude of these deviations will be reasonably small for values of the parameters α and $2h/l$ in the usual practical range.

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REFERENCES

1. E. REISSNER, *A Contribution to the Theory of Elasticity of Non-Isotropic Materials (With Applications to Problems of Bending and Torsion)*, Phil. Mag. *30* (1940), pp. 418 to 427.
2. F. B. HILDEBRAND AND E. REISSNER, *Distribution of Stress in Built-in Beams of Narrow Rectangular Cross Section*, Journ. Appl. Mech. *9* (1942), pp. A-108 to A-116.

SEVEN-POINT LAGRANGIAN INTEGRATION FORMULAS

By G. Blanch and I. Rhodes

Sometimes it is required to find the integral of a function $f(x)$ over an extended region when values of $f(x)$ at equally spaced points (at intervals h, say) are known throughout the region. If one desires the value of the integral from x_1 to x_s say, without requiring the values of the integral at intermediate points, one may construct a polynomial $\varphi_{s-1}(x)$ of degree $(s-1)$ through these points and then integrate the polynomial.

Bickley[1] has given these integration formulas for $\varphi_s(x)$, when $s = 2, 3, 4, 5, 6, 8, 10$, with the remainder terms for the integral of $f(x)$, when the latter is approximated by $\varphi_s(x)$. The formulas become more cumbersome as s increases; and if the integrand $f(x)$ may be approximated by polynomials of degree less than s, when s is large, simpler formulas may be adequate. Simpson's rule (for an odd number of points s) is based on approximating the integrand by a sequence of polynomials of the second degree. In this paper we shall give formulas (not requiring differences) for the integral of $f(x)$, when the latter has been approximated by polynomials of degree six, thus affording greater accuracy than by Simpson's rule; or for comparable accuracy, permitting the evaluation of the integrand at fewer points in the region.

Before giving these formulas, it will be worth while to examine certain remainder terms.

Let

$$\{x_j, x_k; x_{1,\cdots,t}\}$$

denote the integral from x_j to x_k of the polynomial $\varphi(x)$ determined by $x_1 + ih$, $i = 0, 1, \cdots, t-1$; and let

$$R\{x_j, x_k; x_{1,\cdots,t}\}$$

denote the remainder term of the integral of $f(x)$ over the same region, when $f(x)$ is approximated by $\varphi(x)$. Then we may prove[2]:

$$(1)\quad R\{x_1, x_7; x_{1,\cdots,7}\} = -0.0064 \times h^9 f^{(8)}(\xi)$$

$$(2)\quad R\{x_3, x_5; x_{1,\cdots,7}\} = -0.00020 \times h^9 f^{(8)}(\xi)$$

$$(3)\quad R\{x_1, x_2; x_{1,\cdots,7}\} = +0.011 \times h^8 f^{(7)}(\xi)$$

$$(4)\quad |R\{x_1, x_3; x_{1,\cdots,7}\} + R\{x_1, x_4; x_{1,\cdots,7}\} + R\{x_{s-2}, x_s; x_{s-6,\cdots,s}\} + R\{x_{s-3}, x_s; x_{s-6,\cdots,s}\}| < 0.03(s-1) \times |h^9 f^{(8)}(\xi)|$$

[1] W. G. Bickley: "Formulae for Numerical Integration," Math. Gazette, vol. 23 (1939), pp. 352–359.

[2] The remainders (1) and (5) are derived in Milne-Thomson's *The Calculus of Finite Differences*, Sections 7.13 and 7.14. Macmillan, London (1933). All others may be obtained by similar methods.

$$(5)\quad R\{x_1, x_3; x_{1,\cdots,3}\} = -0.016 \times h^9 f^{(8)}(\xi)$$

$$(6)\quad R\{x_4, x_5; x_{1,\cdots,3}\} = +0.0007 \times h^9 f^{(8)}(\xi)$$

$$(7)\quad |R\{x_1, x_6; x_{1,\cdots,3}\}| < 0.014 \times |h^9 f^{(8)}(\xi)|$$

In the above, ξ is some point in the region containing the points specified in the brace, on the left hand side of the equations or inequalities.

If it is assumed that all the values of $h^9 f^{(8)}(\xi)$ are of the same order of magnitude, and that they are of a lower order of magnitude than the values of $h^8 f^{(7)}(\xi)$, then the coefficients in formulas (1) and (3) show the striking fact that the remainder over a region from x_1 to $(x_1 + 6h)$ is numerically less than that over a strip from x_1 to $(x_1 + h)$. Moreover, the remainder in an odd-point formula is least for the middle strip of width $2h$ [as, for example, formula (2)]. It is also worth noting the well known fact that the coefficient in the remainder over an eight-point strip [formula (5)] is more than double that in the seven-point strip [formula (1)].

Of the first three formulas given below, formula (8) is simplest and may be used when there are either an even or an odd number of points in the interval. The remainder term is somewhat larger than that in formulas (9) or (10).

Formula (8): for $s \geq 14$.

$$\begin{aligned}\int_{x_1}^{x_1+(s-1)h} f(x)\,dx = \frac{h}{120960}\,[&36799(f_1 + f_s) + 176648(f_2 + f_{s-1})\\ &+ 54851(f_3 + f_{s-2}) + 177984(f_4 + f_{s-3}) + 89437(f_5 + f_{s-4})\\ &+ 130936(f_6 + f_{s-5}) + 119585(f_7 + f_{s-6})]\\ &+ h(f_8 + f_9 + \cdots + f_{s-7}) + R\end{aligned}$$

where

$$|R| < 0.02(s-1) \times |h^9 f^{(8)}(\xi)|$$

[If $s = 14$, the set of terms $h(f_8 + f_9 + \cdots + f_{s-7})$ should be dropped.]

Method of integration:

$$\frac{1}{2}\left\{\begin{array}{l}\{x_1, x_4; x_{1,\cdots,3}\} + \{x_4, x_5; x_{1,\cdots,3}\} + \{x_5, x_6; x_{2,\cdots,8}\} + \cdots\\ +\{x_{s-3}, x_{s-2}; x_{s-6,\cdots,s}\} + \{x_{s-2}, x_s; x_{s-6,\cdots,s}\} + \{x_1, x_3; x_{1,\cdots,7}\}\\ +\{x_3, x_4; x_{1,\cdots,7}\} + \{x_4, x_5; x_{2,\cdots,8}\} + \cdots + \{x_{s-4}, x_{s-3}; x_{s-6,\cdots,s}\}\\ +\{x_{s-3}, x_s; x_{s-6,\cdots,s}\}\end{array}\right\}$$

Formula (9): for odd number of points $s \geq 19$.

$$\begin{aligned}\int_{x_1}^{x_1+(s-1)h} f(x)\,dx = \frac{h}{3780}\,[&1107(f_1 + f_s) + 5832(f_2 + f_{s-1}) + 729(f_3 + f_{s-1})\\ &+ 7344(f_4 + f_{s-3}) + 734(f_5 + f_{s-4}) + 5760(f_6 + f_{s-5})\\ &+ 2615(f_7 + f_{s-6}) + 4616(f_8 + f_{s-7}) + 3011(f_9 + f_{s-8})\\ &+ 4544(f_{10} + f_{12} + \cdots + f_{s-9})\\ &+ 3016(f_{11} + f_{13} + \cdots + f_{s-10})] + R\end{aligned}$$

where

$$R = -[0.0001(s - 13) + 0.013] \times h^9 f^{(8)}(\xi).$$

[If $s = 19$, the set of terms $3016(f_{11} + f_{12} + \cdots + f_{s-10})$ should be dropped.]

Method of integration:

$$\{x_1, x_7 ; x_{1,\cdots,7}\} + \{x_7, x_9 ; x_{5,\cdots,11}\} + \cdots + \{x_{s-8}, x_{s-6} ; x_{s-10,\cdots,s-4}\} + \{x_{s-6}, x_s ; x_{s-6,\cdots,s}\}.$$

Formula (10): for even number of points $s \geq 22$.

$$\begin{aligned}\int_{x_1}^{x_1+(s-1)h} f(x)\,dx = \frac{h}{241920}\,[&70848(f_1 + f_s) + 373248(f_2 + f_{s-1}) \\ &+ 46656(f_3 + f_{s-2}) + 469825(f_4 + f_{s-3}) + 48695(f_5 + f_{s-4}) \\ &+ 361573(f_6 + f_{s-5}) + 185123(f_7 + f_{s-6}) \\ &+ 264291(f_8 + f_{s-7}) + 234533(f_9 + f_{s-8}) \\ &+ 243639(f_{10} + f_{s-9}) + 241729(f_{11} + f_{s-10})] \\ &+ h(f_{12} + f_{13} + \cdots + f_{s-11}) + R\end{aligned}$$

where

$$|R| < [0.0001(s - 14) + 0.014] \times |h^9 f^{(8)}(\xi)|.$$

[If $s = 22$, the set of terms $(f_{12} + f_{13} + \cdots + f_{s-11})$ should be dropped.]

Method of integration:

$$\frac{1}{2}\left\{\begin{aligned}&\{x_1, x_7 ; x_{1,\cdots,7}\} + \{x_7, x_8 ; x_{4,\cdots,11}\} + \{x_8, x_{10} ; x_{6,\cdots,12}\} \\ &+ \cdots + \{x_{s-8}, x_{s-6} ; x_{s-10,\cdots,s-4}\} + \{x_{s-6}, x_s ; x_{s-6,\cdots,s}\} \\ &+ \{x_1, x_7 ; x_{1,\cdots,7}\} + \{x_7, x_9 ; x_{5,\cdots,11}\} + \cdots + \{x_{s-9}, x_{s-7} ; x_{s-11,\cdots,s-5}\} \\ &+ \{x_{s-7}, x_{s-6} ; x_{s-10,\cdots,s-3}\} + \{x_{s-6}, x_s ; x_{s-6,\cdots,s}\}\end{aligned}\right\}$$

Bickley's formulas previously mentioned may be used for s less than 12. Bickley gave no formulas for the 7th and 9th degree polynomials, however, because these are less accurate than the 6th and 8th degree polynomials respectively. But since it is not always possible to divide the region into the most suitable number of sub-intervals, we shall add below for completeness the integral of 8-, 10-, 12- and 13-point integrands (the last two based on sixth degree polynomials).

Formula (11): $s = 8$.

$$\begin{aligned}\int_{x_1}^{x_8} f(x)\,dx = \frac{7h}{17280}\,[&751(f_1 + f_8) + 3577(f_2 + f_7) \\ &+ 1323(f_3 + f_6) + 2989(f_4 + f_5)] + R\end{aligned}$$

$$R = -0.016 \times h^9 f^{(8)}(\xi).$$

Method of integration:

$$\{x_1, x_8 ; x_{1,\cdots,8}\}$$

Formula (12): $s = 10$.

$$\int_{x_1}^{x_{10}} f(x)\,dx = \frac{9h}{89600}\,[2857(f_1 + f_{10}) + 15741(f_2 + f_9) + 1080(f_3 + f_8) + 19344(f_4 + f_7) + 5778(f_5 + f_6)] + R$$

$$R = -0.012 \times h^{11} f^{(10)}(\xi).$$

Method of integration:

$$\{x_1\,,\ x_{10}\,;\ x_{1,\cdots,10}\}.$$

Formula (13): $s = 12$.

$$\int_{x_1}^{x_{12}} f(x)\,dx = \frac{h}{241920}\,[71199(f_1 + f_{12}) + 370249(f_2 + f_{11}) + 58203(f_3 + f_{10}) + 443133(f_4 + f_9) + 92828(f_5 + f_8) + 294948(f_6 + f_7)] + R$$

$$|R| < 0.02 \times |h^9 f^{(9)}(\xi)|.$$

Method of integration

$$\frac{1}{2}\left\{\begin{matrix}\{x_1\,,\ x_7;\ x_{1,\cdots,7}\} + \{x_7\,,\ x_{12};\ x_{5,\cdots,12}\} + \{x_1\,,\ x_6;\ x_{1,\cdots,6}\} \\ +\ \{x_6\,,\ x_{12};\ x_{6,\cdots,12}\}\end{matrix}\right\}$$

Formula (14): $s = 13$

$$\int_{x_1}^{x_{13}} f(x)\,dx = \frac{h}{140}\,[41(f_1 + 2f_7 + f_{13}) + 216(f_2 + f_6 + f_8 + f_{12}) + 27(f_3 + f_5 + f_9 + f_{11}) + 272(f_4 + f_{10})] + R$$

$$R \doteq -0.013 \times h^9 f^{(9)}(\xi).$$

Method of integration:

$$\{x_1\,,\ x_7\,;\ x_{1,\cdots,7}\} + \{x_7\,,\ x_{13}\,;\ x_{7,\cdots,13}\}$$

As an example, let us integrate $(1/x^2)$ for the ranges indicated below by formulas (8), (9), (10), and (13). A summary of the results follows:

Range in x	Number of points	Type of Formula and Value Obtained	Correct Value
2.0–3.1	12	(13) 0.17741 93556	0.17741 93549
2.0–4.4	25	(8) 0.27272 72751 (9) 0.27272 72733	0.27272 72727
2.0–4.7	28	(8) 0.28723 40451 (10) 0.28723 40431	0.28723 40426

It is thus seen that in the above examples, results obtained by formula (8) are almost as good as those obtained by the more complicated ones.

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A SHORT TABLE OF THE FIRST FIVE ZEROS OF THE TRANSCENDENTAL EQUATION $J_0(x)Y_0(kx) - J_0(kx)Y_0(x) = 0$

By A. N. Lowan and A. Hillman*

The first five zeros of the above equation were computed for $k = 1\frac{1}{2}, 2, 2\frac{1}{2}, 3, 3\frac{1}{2}, 4$. The zeros are given in the accompanying Table I. The products of these zeros by $(k - 1)$ are given in Table II.

Two methods of computation were employed:

(1) The zeros were calculated by means of the asymptotic expansion[1]

$$x_s = b + p/b + (q - p^2)/b^3 + (r - 4pq + 2p^3)/b^5 + \cdots,$$

where x_s is the s^{th} zero,

$$b = s\pi/(k - 1), \qquad p = -1/8k, \qquad q = 100(k^2 - 1)/3(8k)^3(k - 1),$$
$$r = -32(1073)(k^5 - 1)/5(8k)^5(k - 1).$$

TABLE I

The first five zeros of $J_0(x)Y_0(kx) - J_0(kx)Y_0(x) = 0$

	x_1	x_2	x_3	x_4	x_5
$k = 1\frac{1}{2}$	(6.2702)35	12.559781*	18.845147*	25.129431*	31.413277*
$k = 2$	(3.12)3031	(6.27343)6	(9.41820)8	12.561423*	15.703998*
$k = 2\frac{1}{2}$	(2.07)3229	(4.1772)99	(6.27537)0	(8.371674)	10.467233*
$k = 3$	(1.5)48459	(3.129)084	(4.7037)97	(6.27666)5	(7.84873)5
$k = 3\frac{1}{2}$	(1.2)33875	(2.500)165	(3.7608)16	(5.01961)4	(6.27759)2
$k = 4$	(1.0)24421	(2.080)943	(3.132)167	(4.1815)69	(5.2301)48

Note. As $k \to 1$, $(k - 1)x_s \to s\pi$.

The meaning of the asterisks and of the parentheses is explained on the previous page.

(2) Tables of $J_0(x)Y_0(kx) - Y_0(x)J_0(kx)$ in the vicinity of the roots were prepared using the British Association Tables.[2] The zeros were then recomputed by inverse interpolation using the formula[3]

$$p = m - m(m - 1)[D_1/2! + (m - 2)D_2/3!$$
$$+ (m - 2)(m - 3)D_3/4! + \cdots + (1 - 2m)(D_1/2)^2 + \cdots]$$

* Members of the National Bureau of Standards, Mathematical Tables Project.

[1] Prof. James McMahon: "On the Roots of the Bessel and Certain Related Functions." (Annals of Math., Jan. 1895)

[2] British Association Mathematical Tables, Vol. VI, Part I.

[3] Harold T. Davis: "Tables of the Higher Mathematical Functions," page 80.

where

$$m = [f(x + pd) - f(x)]/\Delta f(x), \qquad Di = \Delta^{i+1} f(x)/\Delta f(x).$$

Both methods were used for all but the seven zeros for which kx_s is greater than 25. In these seven cases, the values were checked by duplicate calculation

TABLE II

Values of $(k - 1)x_s$ *for* $s = 1, 2, 3, 4, 5$

	$(k-1)\,x_1$	$(k-1)\,x_2$	$(k-1)\,x_3$	$(k-1)\,x_4$	$(k-1)\,x_5$
$k = 1$	3.141593	6.283185	9.424778	12.566371	15.707963
$k = 1\frac{1}{2}$	3.135118	6.279890	9.422574	12.564715	15.706638
$k = 2$	3.123031	6.273436	9.418208	12.561423	15.703998
$k = 2\frac{1}{2}$	3.109843	6.265948	9.413055	12.557511	15.700850
$k = 3$	3.096918	6.258168	9.407594	12.553329	15.697469
$k = 3\frac{1}{2}$	3.084687	6.250412	9.402040	12.549034	15.693979
$k = 4$	3.073264	6.242830	9.396500	12.544708	15.690443

using the asymptotic expansion. The roots computed by inverse interpolation were checked by direct interpolation using the Gregory-Newton formula

$$f(x + pd) = f(x) + p\Delta f(x) + [p(p - 1)/2!]\Delta^2 f(x) + \cdots$$

The "starred" values were computed by method (1) only, all other values were computed by both methods; the digits enclosed in parentheses are the values obtained by method (1). All zeros are correct to within one unit in the last decimal place. Table (2) is more convenient than table (1) for purposes of interpolation for values of k between 1 and 4. Values obtained by linear interpolation are correct to within two units in the fourth place.

TABLE OF COEFFICIENTS FOR INVERSE INTERPOLATION WITH CENTRAL DIFFERENCES

By Herbert E. Salzer[1]

Although many tables have been published to facilitate direct interpolation, up to now there has been no table to aid in the more cumbersome task of inverse interpolation. A very convenient formula is given in H. T. Davis "Tables of the Higher Mathematical Functions," vol. I, pp. 82–83. It is obtained by applying Lagrange's theorem on inversion to the Everett interpolation formula. The expression given by Davis was checked and found to be entirely correct except for an obvious printing error, where the coefficient of $\frac{\delta_0^2}{\Delta}\left(\frac{\delta_1^2}{\Delta}\right)^2 \Big/ 36$ should begin with a 1 instead of a 2.

If a given function has a value y lying between y_0 and y_1 corresponding to arguments x_0 and x_1 respectively, the problem of inverse interpolation is to find x such that the function of x is equal to y. When there is a uniform interval h for the argument, if Δ denotes $y_1 - y_0$, m denotes $(y - y_0)/\Delta$, the problem is: given m, to find p such that $x = x_0 + ph$. The problem is restricted to the case where the function is monotonic between x_0 and x_1, so that $0 < m < 1$. Also, this formula for inverse interpolation is designed to obtain only that value of x within the interval $[x_0, x_1]$, i.e. $0 < p < 1$.

Employing the central difference notation of δ_0^r and δ_1^r to indicate the values of the interlinear differences in the rth column on the same row as y_0 and y_1, respectively, Davis' formula reads:

$$
\begin{aligned}
p = m &+ \frac{m(1-m)(2-m)}{6}\frac{\delta_0^2}{\Delta} + \frac{m(1-m^2)}{6}\frac{\delta_1^2}{\Delta} \\
&- \frac{m(1-m)(2-m)(3+2m-m^2)}{120}\frac{\delta_0^4}{\Delta} - \frac{m(1-m^2)(4-m^2)}{120}\frac{\delta_1^4}{\Delta} \\
&+ \frac{m(1-m)(2-m)(2-6m+3m^2)}{36}\left(\frac{\delta_0^2}{\Delta}\right)^2 + \frac{m(1-m^2)(1-3m^2)}{36}\left(\frac{\delta_1^2}{\Delta}\right)^2 \\
&+ \frac{m(4-9m-4m^2+15m^3-6m^4)}{36}\frac{\delta_0^2}{\Delta}\frac{\delta_1^2}{\Delta} \\
&+ \frac{m(1-m)(2-m)(3+2m-m^2)(8+2m-m^2)}{5040}\frac{\delta_0^6}{\Delta} \\
&+ \frac{m(1-m^2)(4-m^2)(9-m^2)}{5040}\frac{\delta_1^6}{\Delta} \\
&- \frac{m(1-m)(2-m)(3-6m-5m^2+8m^3-2m^4)}{180}\frac{\delta_0^2}{\Delta}\frac{\delta_0^4}{\Delta}
\end{aligned}
$$

[1] Mathematical Tables Project, National Bureau of Standards.

$$- \frac{m(1-m^2)(2-7m^2+2m^4)}{180} \frac{\delta_1^2}{\Delta} \frac{\delta_1^4}{\Delta}$$

$$- \frac{m(16-36m-24m^2+75m^3-18m^4-21m^5+8m^6)}{720} \frac{\delta_0^2}{\Delta} \frac{\delta_1^4}{\Delta}$$

$$- \frac{m(12-15m-44m^2+50m^3+24m^4-35m^5+8m^6)}{720} \frac{\delta_1^2}{\Delta} \frac{\delta_0^4}{\Delta}$$

$$+ \frac{m(2-12m+15m^2+15m^3-42m^4+28m^5-6m^6)}{36} \left(\frac{\delta_0^2}{\Delta}\right)^2 \frac{\delta_1^2}{\Delta}$$

$$+ \frac{m(1-3m-5m^2+15m^3-14m^4+6m^5)}{36} \left(\frac{\delta_1^2}{\Delta}\right)^2 \frac{\delta_0^2}{\Delta}$$

$$+ \frac{m(1-m)(2-m)(4-30m+63m^2-48m^3+12m^4)}{216} \left(\frac{\delta_0^2}{\Delta}\right)^3$$

$$+ \frac{m(1-m^2)(1-9m^2+12m^4)}{216} \left(\frac{\delta_1^2}{\Delta}\right)^3 + \cdots$$

It is natural to refer to the coefficients of $\frac{\delta_0^2}{\Delta}$ and $\frac{\delta_1^2}{\Delta}$ as belonging to second order terms, the coefficients of $\frac{\delta_0^4}{\Delta}, \frac{\delta_1^4}{\Delta}, \left(\frac{\delta_0^2}{\Delta}\right)^2, \left(\frac{\delta_1^2}{\Delta}\right)^2$ and $\frac{\delta_0^2}{\Delta}\frac{\delta_1^2}{\Delta}$ as belonging to fourth order terms and the coeficients of $\frac{\delta_0^6}{\Delta}, \frac{\delta_1^6}{\Delta}, \frac{\delta_0^2}{\Delta}\frac{\delta_0^4}{\Delta}, \frac{\delta_1^2}{\Delta}\frac{\delta_1^4}{\Delta}, \frac{\delta_0^2}{\Delta}\frac{\delta_1^4}{\Delta}, \frac{\delta_1^2}{\Delta}\frac{\delta_0^4}{\Delta}, \left(\frac{\delta_0^2}{\Delta}\right)^2\frac{\delta_1^2}{\Delta}, \left(\frac{\delta_1^2}{\Delta}\right)^2\frac{\delta_0^2}{\Delta}, \left(\frac{\delta_0^2}{\Delta}\right)^3$ and $\left(\frac{\delta_1^2}{\Delta}\right)^3$ as belonging to sixth order terms. It is worth noting that (similar to the use of the Everett formula for direct interpolation) the use of this formula including terms of the nth order really gives accuracy as far as the $(n+1)$th order. The following tables give the five fourth order coefficients for $m = 0(.001)1.000$; to 10 decimals (good to about a unit in the last decimal), and also the ten sixth order coefficients for $m = 0(.1)1.0$; exact values. A condensed one-page table of fourth order coeficients at intervals of 0.1 is also given for convenience in cases where a larger table is unnecessary.

It was not deemed necessary to tabulate the two second order coefficients because they are simply $x(1-x^2)/6$ for $x = m$ in the case of $\frac{\delta_1^2}{\Delta}$ and for $x = 1 - m$ in the case of $\frac{\delta_0^2}{\Delta}$, and exact values of the function $x(1-x^2)/6$ are tabulated at intervals of 0.0001 by the Mathematical Tables Project in the Table of Lagrangian Interpolation Coefficients among the four-point interpolants. For x (there "p") ranging from 0 to 1, it is the function $-A_2$. Linear interpolation in that table will furnish 9 decimals. However, for great accuracy, it is just about as expedient to compute directly $x(1-x^2)/6$ for $x = m$ and $x = 1 - m$ on a good calculating machine, as to perform two interpolations in a table. One

suggestion is to obtain $m(1 - m)/6$ and then to multiply by both $2 - m$ and $1 + m$ to obtain the coefficients for both $\frac{\delta_0^2}{\Delta}$ and $\frac{\delta_1^2}{\Delta}$.

Upper bounds for the errors in x, due to the neglect of higher order terms in inverse interpolation by this formula, are as follows:

Linear inverse interpolation has an upper bound for the error due to neglect of second order terms equal to $\frac{h}{8}\frac{\delta^2}{\Delta}$,† provided that fourth order terms are negligible.

Quadratic inverse interpolation with omission of all fourth order terms has an upper bound for the error in x equal to $\frac{h}{40}\left[\left(\frac{\delta^2}{\Delta}\right)^2 + \frac{\delta^4}{\Delta}\right]$, if sixth order terms can be neglected.

Fourth order inverse interpolation, i.e. use of the main part of this table, has an upper bound for the error due to sixth order terms equal to $h \times 0.01\left[\left(\frac{\delta^2}{\Delta}\right)^3 + \frac{\delta^2}{\Delta}\frac{\delta^4}{\Delta} + \frac{\delta^6}{\Delta}\right]$ or, even more precisely, $h \times \left\{0.01\left[\left(\frac{\delta^2}{\Delta}\right)^3 + \frac{\delta^2}{\Delta}\frac{\delta^4}{\Delta}\right] + 0.005\,\frac{\delta^6}{\Delta}\right\}$, on the assumption that eighth order terms are relatively small.

In cases where this entire table can be used, i.e. for inverse interpolation including sixth order terms, it seems reasonable to expect the error due to the omission of eighth order terms to be of the order of, or even much less than $h \times 0.01\left[\left(\frac{\delta^2}{\Delta}\right)^4 + \left(\frac{\delta^2}{\Delta}\right)^2\frac{\delta^4}{\Delta} + \left(\frac{\delta^4}{\Delta}\right)^2 + \frac{\delta^6}{\Delta}\frac{\delta^2}{\Delta} + \frac{\delta^8}{\Delta}\right]$, assuming that tenth order terms may be disregarded. Due to the prohibitive amount of labor that would be necessary, this last estimate has not been proven, but merely inferred by comparison with the previous upper bounds, all of which have been proven.

† The expression $\frac{\delta^i}{\Delta}$ denotes the larger of the *absolute* values of $\frac{\delta_0^i}{\Delta}$ and $\frac{\delta_1^i}{\Delta}$.

TABLE IA: SHORT TABLE OF COEFFICIENTS OF FOURTH ORDER TERMS

$$x = x_o + ph;\ [F(x) - F(x_o)]/\Delta = m;\ p = m + \tfrac{1}{2}(1 - m)\,[1 - (1 - m)^2]\,\frac{\delta_0^2}{\Delta}$$
$$+ \tfrac{1}{2}[m\,(1 - m^2)]\,\frac{\delta_1^2}{\Delta} + A(m)\,\frac{\delta_0^4}{\Delta} + B(m)\,\frac{\delta_1^4}{\Delta} + C(m)\left(\frac{\delta_0^2}{\Delta}\right)^2 + D(m)\left(\frac{\delta_1^2}{\Delta}\right)^2$$
$$+ E(m)\,\frac{\delta_0^2}{\Delta}\,\frac{\delta_1^2}{\Delta} + \text{Sixth Order Terms} + \cdots$$

m	$A(m)$	$B(m)$	$C(m)$	$D(m)$	$E(m)$	m
.0	.00000 000	.00000 000	.00000 00000	.00000 00000	.00000 00000	.0
.1	−.00454 575	−.00329 175	.00679 25	.00266 75	.00854	.1
.2	−.00806 4	−.00633 6	.00736	.00469 33333̇	.01194 66666̇	.2
.3	−.01044 225	−.00889 525	.00466 08333̇	.00553 58333̇	.01080 33333̇	.3
.4	−.01164 8	−.01075 2 ·	.00085 33333̇	.00485 33333̇	.00629 33333̇	.4
.5	−.01171 875	−.01171 875	−.00260 41666̇	.00260 41666̇	.00000 00000	.5
.6	−.01075 2	−.01164 8	−.00485 33333̇	−.00085 33333̇	−.00629 33333̇	.6
.7	−.00889 525	−.01044 225	−.00553 58333̇	−.00466 08333̇	−.01080 33333̇	.7
.8	−.00633 6	−.00806 4	−.00469 33333̇	−.00736	−.01194 66666̇	.8
.9	−.00329 175	−.00454 575	−.00266 75	−.00679 25	−.00854	.9
1.0	.00000 .000	.00000 000	.00000 00000	.00000 00000	.00000 00000	1.0

TABLE I: COEFFICIENTS OF FOURTH ORDER TERMS

$$x = x_0 + ph;\ [F(x) - F(x_0)]/\Delta = m;\ p = m + \tfrac{1}{6}(1-m)[1-(1-m)^2]\frac{\delta_0^2}{\Delta} + \tfrac{1}{6}[m(1-m^2)]\frac{\delta_1^2}{\Delta}$$

$$+ A(m)\frac{\delta_0^4}{\Delta} + B(m)\frac{\delta_1^4}{\Delta} + C(m)\left(\frac{\delta_0^2}{\Delta}\right)^2 + D(m)\left(\frac{\delta_1^2}{\Delta}\right)^2 + E(m)\frac{\delta_0^2}{\Delta}\frac{\delta_1^2}{\Delta} + \text{Sixth Order Terms} + \ldots$$

m	$-A(m)$	$-B(m)$	$-C(m)$	$D(m)$	$E(m)$	
0.000	.00000 00000	.00000 00000	.00000 00000	.00000 00000	.00000 00000	1.000
.001	.00004 99583	.00003 33333	−.00011 06118	.00002 77777	.00011 08610	0.999
.002	.00009 98330	.00006 66663	.00022 02280	.00005 55547	.00022 12213	.998
.003	.00014 96239	.00009 99989	.00032 88528	.00008 33303	.00033 10804	.997
.004	.00019 93307	.00013 33307	.00043 64906	.00011 11040	.00044 04374	.996
0.005	.00024 89532	.00016 66615	−.00054 31456	.00013 88750	.00054 92919	0.995
.006	.00029 84911	.00019 99910	.00064 88221	.00016 66427	.00065 76432	.994
.007	.00034 79441	.00023 33190	.00075 35245	.00019 44063	.00076 54907	.993
.008	.00039 73122	.00026 66453	.00085 72570	.00022 21653	.00087 28337	.992
.009	.00044 65949	.00029 99696	.00096 00238	.00024 99190	.00097 96717	.991
0.010	.00049 57921	.00033 32917	−.00106 18292	.00027 76667	.00108 60041	0.990
.011	.00054 49035	.00036 66112	.00116 26774	.00030 54077	.00119 18304	.989
.012	.00059 39289	.00039 99280	.00126 25727	.00033 31414	.00129 71499	.988
.013	.00064 28680	.00043 32418	.00136 15193	.00036 08670	.00140 19622	.987
.014	.00069 17206	.00046 65523	.00145 95214	.00038 85840	.00150 62666	.986
0.015	.00074 04865	.00049 98594	−.00155 65831	.00041 62917	.00161 00626	0.985
.016	.00078 91654	.00053 31627	.00165 27088	.00044 39894	.00171 33498	.984
.017	.00083 77571	.00056 64620	.00174 79025	.00047 16765	.00181 61276	.983
.018	.00088 62614	.00059 97570	.00184 21684	.00049 93522	.00191 83954	.982
.019	.00093 46780	.00063 30476	.00193 55107	.00052 70159	.00202 01529	.981
0.020	.00098 30066	.00066 63334	−.00202 79336	.00055 46669	.00212 13995	0.980
.021	.00103 12472	.00069 96142	.00211 94411	.00058 23047	.00222 21347	.979
.022	.00107 93994	.00073 28897	.00221 00375	.00060 99284	.00232 23581	.978
.023	.00112 74630	.00076 61598	.00229 97268	.00063 75375	.00242 20692	.977
.024	.00117 54378	.00079 94241	.00238 85131	.00066 51313	.00252 12676	.976
0.025	.00122 33235	.00083 26824	−.00247 64006	.00069 27091	.00261 99528	0.975
.026	.00127 11199	.00086 59344	.00256 33932	.00072 02703	.00271 81244	.974
.027	.00131 88269	.00089 91800	.00264 94953	.00074 78142	.00281 57820	.973
.028	.00136 64441	.00093 24188	.00273 47107	.00077 53401	.00291 29252	.972
.029	.00141 39714	.00096 56506	.00281 90435	.00080 28474	.00300 95536	.971
0.030	.00146 14085	.00099 88752	−.00290 24979	.00083 03354	.00310 56668	0.970
.031	.00150 87553	.00103 20923	.00298 50778	.00085 78034	.00320 12644	.969
.032	.00155 60114	.00106 53016	.00306 67872	.00088 52508	.00329 63460	.968
.033	.00160 31767	.00109 85030	.00314 76303	.00091 26769	.00339 09113	.967
.034	.00165 02510	.00113 16960	.00322 76110	.00094 00811	.00348 49599	.966
0.035	.00169 72340	.00116 48806	−.00330 67333	.00096 74627	.00357 84915	0.965
.036	.00174 41255	.00119 80565	.00338 50012	.00099 48210	.00367 15058	.964
.037	.00179 09253	.00123 12234	.00346 24187	.00102 21554	.00376 40023	.963
.038	.00183 76332	.00126 43810	.00353 89898	.00104 94653	.00385 59809	.962
.039	.00188 42490	.00129 75291	.00361 47184	.00107 67499	.00394 74412	.961
0.040	.00193 07725	.00133 06675	−.00368 96085	.00110 40085	.00403 83829	0.960
.041	.00197 72034	.00136 37959	.00376 36641	.00113 12407	.00412 88058	.959
.042	.00202 35416	.00139 69141	.00383 68890	.00115 84456	.00421 87094	.958
.043	.00206 97868	.00143 00218	.00390 92872	.00118 56226	.00430 80937	.957
.044	.00211 59388	.00146 31187	.00398 08627	.00121 27711	.00439 69582	.956
0.045	.00216 19974	.00149 62047	−.00405 16193	.00123 98904	.00448 53028	0.955
.046	.00220 79625	.00152 92794	.00412 15609	.00126 69798	.00457 31273	.954
.047	.00225 38338	.00156 23426	.00419 06914	.00129 40388	.00466 04313	.953
.048	.00229 96111	.00159 53941	.00425 90147	.00132 10666	.00474 72147	.952
.049	.00234 52941	.00162 84336	.00432 65347	.00134 80625	.00483 34773	.951
0.050	.00239 08828	.00166 14609	−.00439 32552	.00137 50260	.00491 92188	0.950
	$-B(m)$	$-A(m)$	$D(m)$	$-C(m)$	$-E(m)$	m

TABLE I: COEFFICIENTS OF FOURTH ORDER TERMS—*Continued*

$$x = x_0 + ph;\ [F(x) - F(x_0)]/\Delta = m;\ p = m + \tfrac{1}{2}(1-m)[1-(1-m)^2]\frac{\delta_0^2}{\Delta} + \tfrac{1}{3}[m(1-m^2)]\frac{\delta}{\Delta}$$

$$+ A(m)\frac{\delta_0^4}{\Delta} + B(m)\frac{\delta_1^4}{\Delta} + C(m)\left(\frac{\delta_0^2}{\Delta}\right)^2 + D(m)\left(\frac{\delta_1^2}{\Delta}\right)^2 + E(m)\frac{\delta_0^2}{\Delta}\frac{\delta_1^2}{\Delta} + \text{Sixth Order Terms} + \ldots$$

m	$-A(m)$	$-B(m)$	$-C(m)$	$D(m)$	$E(m)$	
0.050	.00239 08828	.00166 14609	−.00439 32552	.00137 50260	.00491 92188	0.950
.051	.00243 63769	.00169 44758	.00445 91801	.00140 19564	.00500 44390	.949
.052	.00248 17761	.00172 74778	.00452 43132	.00142 88530	.00508 91378	.948
.053	.00252 70804	.00176 04669	.00458 86583	.00145 57152	.00517 33150	.947
.054	.00257 22895	.00179 34428	.00465 22193	.00148 25423	.00525 69704	.946
.055	.00261 74031	.00182 64052	−.00471 50000	.00150 93336	.00534 01039	0.945
.056	.00266 24212	.00185 93539	.00477 70042	.00153 60886	.00542 27152	.944
.057	.00270 73434	.00189 22886	.00483 82356	.00156 28065	.00550 48044	.943
.058	.00275 21697	.00192 52091	.00489 86982	.00158 94867	.00558 63711	.942
.059	.00279 68998	.00195 81152	.00495 83955	.00161 61286	.00566 74154	.941
0.060	.00284 15335	.00199 10065	−.00501 73315	.00164 27315	.00574 79371	0.940
.061	.00288 60707	.00202 38828	.00507 55098	.00166 92947	.00582 79360	.939
.062	.00293 05110	.00205 67440	.00513 29342	.00169 58177	.00590 74121	.938
.063	.00297 48545	.00208 95896	.00518 96085	.00172 22997	.00598 63653	.937
.064	.00301 91008	.00212 24196	.00524 55363	.00174 87401	.00606 47955	.936
0.065	.00306 32497	.00215 52336	−.00530 07214	.00177 51384	.00614 27027	0.935
.066	.00310 73012	.00218 80314	.00535 51676	.00180 14937	.00622 00867	.934
.067	.00315 12549	.00222 08128	.00540 88784	.00182 78055	.00629 69476	.933
.068	.00319 51108	.00225 35774	.00546 18576	.00185 40732	.00637 32853	.932
.069	.00323 88686	.00228 63252	.00551 41089	.00188 02960	.00644 90996	.931
0.070	.00328 25281	.00231 90557	−.00556 56359	.00190 64734	.00652 43907	0.930
.071	.00332 60892	.00235 17687	.00561 64423	.00193 26047	.00659 91585	.929
.072	.00336 95516	.00238 44641	.00566 65318	.00195 86892	.00667 34030	.928
.073	.00341 29153	.00241 71416	.00571 59080	.00198 47264	.00674 71241	.927
.074	.00345 61799	.00244 98008	.00576 45745	.00201 07156	.00682 03219	.926
0.075	.00349 93455	.00248 24417	−.00581 25350	.00203 66561	.00689 29964	0.925
.076	.00354 24116	.00251 50638	.00585 97931	.00206 25473	.00696 51477	.924
.077	.00358 53783	.00254 76670	.00590 63523	.00208 83886	.00703 67756	.923
.078	.00362 82452	.00258 02511	.00595 22163	.00211 41793	.00710 78804	.922
.079	.00367 10123	.00261 28157	.00599 73887	.00213 99188	.00717 84620	.921
0.080	.00371 36794	.00264 53606	−.00604 18731	.00216 56064	.00724 85205	0.920
.081	.00375 62462	.00267 78857	.00608 56729	.00219 12416	.00731 80560	.919
.082	.00379 87126	.00271 03906	.00612 87919	.00221 68236	.00738 70685	.918
.083	.00384 10785	.00274 28750	.00617 12335	.00224 23519	.00745 55581	.917
.084	.00388 33436	.00277 53389	.00621 30012	.00226 78258	.00752 35250	.916
0.085	.00392 55078	.00280 77818	−.00625 40987	.00229 32448	.00759 09691	0.915
.086	.00396 75710	.00284 02035	.00629 45293	.00231 86080	.00765 78906	.914
.087	.00400 95329	.00287 26039	.00633 42968	.00234 39150	.00772 42897	.913
.088	.00405 13934	.00290 49826	.00637 34045	.00236 91651	.00779 01664	.912
.089	.00409 31524	.00293 73395	.00641 18559	.00239 43577	.00785 55209	.911
0.090	.00413 48095	.00296 96742	−.00644 96546	.00241 94921	.00792 03533	0.910
.091	.00417 63648	.00300 19865	.00648 68040	.00244 45677	.00798 46638	.909
.092	.00421 78180	.00303 42763	.00652 33075	.00246 95839	.00804 84526	.908
.093	.00425 91690	.00306 65431	.00655 91687	.00249 45401	.00811 17197	.907
.094	.00430 04176	.00309 87868	.00659 43910	.00251 94356	.00817 44654	.906
0.095	.00434 15637	.00313 10072	−.00662 89779	.00254 42698	.00823 66898	0.905
.096	.00438 26070	.00316 32039	.00666 29327	.00256 90421	.00829 83931	.904
.097	.00442 35475	.00319 53769	.00669 62589	.00259 37519	.00835 95756	.903
.098	.00446 43849	.00322 75257	.00672 89599	.00261 83986	.00842 02375	.902
.099	.00450 51191	.00325 96501	.00676 10392	.00264 29815	.00848 03788	.901
0.100	.00454 57500	.00329 17500	−.00679 25000	.00266 75000	.00854 00000	0.900
	$-B(m)$	$-A(m)$	$D(m)$	$-C(m)$	$-E(m)$	m

TABLE I: COEFFICIENTS OF FOURTH ORDER TERMS—*Continued*

$$x = x_0 + ph;\ [F(x) - F(x_0)]/\Delta = m;\ p = m + \tfrac{1}{2}(1-m)[1-(1-m)^2]\frac{\delta_0^2}{\Delta} + \tfrac{1}{2}[m(1-m^2)]\frac{\delta_1^2}{\Delta}$$
$$+ A(m)\frac{\delta_0^4}{\Delta} + B(m)\frac{\delta_1^4}{\Delta} + C(m)\left(\frac{\delta_0^2}{\Delta}\right)^2 + D(m)\left(\frac{\delta_1^2}{\Delta}\right)^2 + E(m)\frac{\delta_0^2}{\Delta}\frac{\delta_1^2}{\Delta} + \text{Sixth order terms} + \ldots$$

m	$-A(m)$	$-B(m)$	$-C(m)$	$D(m)$	$E(m)$	
0.100	.00454 57500	.00329 17500	−.00679 25000	.00266 75000	.00854 00000	0.900
.101	.00458 62774	.00332 38250	.00682 33458	.00269 19535	.00859 91011	.899
.102	.00462 67011	.00335 58750	.00685 35801	.00271 63414	.00865 76825	.898
.103	.00466 70211	.00338 78996	.00688 32060	.00274 06631	.00871 57444	.897
.104	.00470 72370	.00341 98987	.00691 22271	.00276 49179	.00877 32870	.896
0.105	.00474 73489	.00345 18720	−.00694 06466	.00278 91052	.00883 03106	0.895
.106	.00478 73565	.00348 38192	.00696 84680	.00281 32245	.00888 68155	.894
.107	.00482 72596	.00351 57401	.00699 56945	.00283 72751	.00894 28019	.893
.108	.00486 70583	.00354 76344	.00702 23294	.00286 12564	.00899 82702	.892
.109	.00490 67522	.00357 95020	.00704 83761	.00288 51679	.00905 32205	.891
0.110	.00494 63412	.00361 13425	−.00707 38379	.00290 90088	.00910 76533	0.890
.111	.00498 58252	.00364 31558	.00709 87181	.00293 27785	.00916 15688	.889
.112	.00502 52041	.00367 49415	.00712 30200	.00295 64766	.00921 49674	.888
.113	.00506 44777	.00370 66995	.00714 67468	.00298 01024	.00926 78493	.887
.114	.00510 36459	.00373 84295	.00716 99018	.00300 36552	.00932 02150	.886
0.115	.00514 27085	.00377 01312	−.00719 24884	.00302 71345	.00937 20647	0.885
.116	.00518 16653	.00380 18044	.00721 45096	.00305 05396	.00942 33987	.884
.117	.00522 05163	.00383 34488	.00723 59689	.00307 38700	.00947 42176	.883
.118	.00525 92613	.00386 50643	.00725 68694	.00309 71251	.00952 45215	.882
.119	.00529 79001	.00389 66506	.00727 72143	.00312 03043	.00957 43109	.881
0.120	.00533 64326	.00392 82074	−.00729 70069	.00314 34069	.00962 35861	0.880
.121	.00537 48588	.00395 97344	.00731 62504	.00316 64325	.00967 23476	.879
.122	.00541 31783	.00399 12316	.00733 49480	.00318 93803	.00972 05958	.878
.123	.00545 13912	.00402 26985	.00735 31028	.00321 22498	.00976 83309	.877
.124	.00548 94973	.00405 41350	.00737 07181	.00323 50404	.00981 55535	.876
0.125	.00552 74963	.00408 55408	−.00738 77970	.00325 77515	.00986 22640	0.875
.126	.00556 53883	.00411 69156	.00740 43428	.00328 03825	.00990 84627	.874
.127	.00560 31731	.00414 82594	.00742 03584	.00330 29329	.00995 41502	.873
.128	.00564 08505	.00417 95717	.00743 58472	.00332 54020	.00999 93268	.872
.129	.00567 84204	.00421 08523	.00745 08122	.00334 77893	.01004 39930	.871
0.130	.00571 58827	.00424 21011	−.00746 52566	.00337 00941	.01008 81493	0.870
.131	.00575 32372	.00427 33177	.00747 91835	.00339 23160	.01013 17961	.869
.132	.00579 04839	.00430 45020	.00749 25960	.00341 44542	.01017 49338	.868
.133	.00582 76225	.00433 56536	.00750 54972	.00343 65083	.01021 75630	.867
.134	.00586 46530	.00436 67724	.00751 78902	.00345 84777	.01025 96841	.866
0.135	.00590 15753	.00439 78580	−.00752 97781	.00348 03617	.01030 12977	0.865
.136	.00593 83892	.00442, 89104	.00754 11640	.00350 21598	.01034 24042	.864
.137	.00597 50946	.00445 99291	.00755 20509	.00352 38715	.01038 30041	.863
.138	.00601 16913	.00449 09141	.00756 24419	.00354 54961	.01042 30980	.862
.139	.00604 81793	.00452 18649	.00757 23401	.00356 70331	.01046 26863	.861
0.140	.00608 45585	.00455 27815	−.00758 17485	.00358 84819	.01050 17696	0.860
.141	.00612 08286	.00458 36635	.00759 06702	.00360 98419	.01054 03484	.859
.142	.00615 69897	.00461 45108	.00759 91081	.00363 11126	.01057 84233	.858
.143	.00619 30415	.00464 53230	.00760 70653	.00365 22934	.01061 59948	.857
.144	.00622 89839	.00467 61000	.00761 45447	.00367 33838	.01065 30635	.856
0.145	.00626 48169	.00470 68414	−.00762 15495	.00369 43831	.01068 96299	0.855
.146	.00630 05403	.00473 75472	.00762 80826	.00371 52909	.01072 56946	.854
.147	.00633 61541	.00476 82169	.00763 41469	.00373 61065	.01076 12581	.853
.148	.00637 16580	.00479 88504	.00763 97455	.00375 68294	.01079 63211	.852
.149	.00640 70520	.00482 94475	.00764 48813	.00377 74590	.01083 08842	.851
0.150	.00644 23359	.00486 00078	−.00764 95573	.00379 79948	.01086 49479	0.850
	$-B(m)$	$-A(m)$	$D(m)$	$-C(m)$	$-E(m)$	m

TABLE I: COEFFICIENTS OF FOURTH ORDER TERMS—*Continued*

$$x = x_0 + ph;\ [F(x) - F(x_0)]/\Delta = m;\ p = m + \tfrac{1}{2}(1-m)[1-(1-m)^2]\frac{\delta_0^2}{\Delta} + \tfrac{1}{2}[m(1-m^2)]\frac{\delta_1^2}{\Delta}$$

$$+ A(m)\frac{\delta_0^4}{\Delta} + B(m)\frac{\delta_1^4}{\Delta} + C(m)\left(\frac{\delta_0^2}{\Delta}\right)^2 + D(m)\left(\frac{\delta_1^2}{\Delta}\right)^2 + E(m)\frac{\delta_0^2}{\Delta}\frac{\delta_1^2}{\Delta} + \text{Sixth Order Terms} + \ldots$$

m	$-A(m)$	$-B(m)$	$-C(m)$	$D(m)$	$E(m)$	
0.150	.00644 23359	.00486 00078	−.00764 95573	.00379 79948	.01086 49479	0.850
.151	.00647 75098	.00489 05312	.00765 37764	.00381 84362	.01089 85129	.849
.152	.00651 25733	.00492 10175	.00765 75416	.00383 87827	.01093 15797	.848
.153	.00654 75265	.00495 14663	.00766 08557	.00385 90338	.01096 41490	.847
.154	.00658 23692	.00498 18775	.00766 37218	.00387 91888	.01099 62214	.846
0.155	.00661 71014	.00501 22508	−.00766 61427	.00389 92472	.01102 77976	0.845
.156	.00665 17228	.00504 25859	.00766 81214	.00391 92085	.01105 88781	.844
.157	.00668 62334	.00507 28827	.00766 96607	.00393 90721	.01108 94637	.843
.158	.00672 06332	.00510 31409	.00767 07636	.00395 88375	.01111 95549	.842
.159	.00675 49219	.00513 33602	.00767 14329	.00397 85041	.01114 91525	.841
0.160	.00678 90995	.00516 35405	−.00767 16715	.00399 80715	.01117 82571	0.840
.161	.00682 31659	.00519 36814	.00767 14822	.00401 75390	.01120 68693	.839
.162	.00685 71210	.00522 37828	.00767 08680	.00403 69061	.01123 49900	.838
.163	.00689 09647	.00525 38444	.00766 98316	.00405 61723	.01126 26196	.837
.164	.00692 46968	.00528 38660	.00766 83759	.00407 53371	.01128 97591	.836
0.165	.00695 83173	.00531 38473	−.00766 65037	.00409 43998	.01131 64089	0.835
.166	.00699 18261	.00534 37881	.00766 42180	.00411 33601	.01134 25699	.834
.167	.00702 52231	.00537 36881	.00766 15214	.00413 22173	.01136 82428	.833
.168	.00705 85082	.00540 35472	.00765 84167	.00415 09710	.01139 34283	.832
.169	.00709 16813	.00543 33651	.00765 49069	.00416 96205	.01141 81271	.831
0.170	.00712 47422	.00546 31415	−.00765 09946	.00418 81655	.01144 23399	0.830
.171	.00715 76910	.00549 28763	.00764 66827	.00420 66053	.01146 60675	.829
.172	.00719 05274	.00552 25691	.00764 19739	.00422 49394	.01148 93107	.828
.173	.00722 32515	.00555 22198	.00763 68711	.00424 31673	.01151 20701	.827
.174	.00725 58631	.00558 18281	.00763 13768	.00426 12886	.01153 43466	.826
0.175	.00728 83621	.00561 13938	−.00762 54940	.00427 93026	.01155 61409	0.825
.176	.00732 07484	.00564 09166	.00761 92253	.00429 72089	.01157 74539	.824
.177	.00735 30220	.00567 03963	.00761 25734	.00431 50069	.01159 82862	.823
.178	.00738 51827	.00569 98327	.00760 55412	.00433 26962	.01161 86386	.822
.179	.00741 72306	.00572 92256	.00759 81313	.00435 02762	.01163 85120	.821
0.180	.00744 91654	.00575 85746	−.00759 03464	.00436 77464	.01165 79072	0.820
.181	.00748 09871	.00578 78797	.00758 21892	.00438 51064	.01167 68249	.819
.182	.00751 26956	.00581 71404	.00757 36624	.00440 23555	.01169 52660	.818
.183	.00754 42908	.00584 63567	.00756 47688	.00441 94934	.01171 32313	.817
.184	.00757 57726	.00587 55282	.00755 55109	.00443 65195	.01173 07216	.816
0.185	.00760 71411	.00590 46548	−.00754 58914	.00445 34333	.01174 77378	0.815
.186	.00763 83960	.00593 37362	.00753 59130	.00447 02344	.01176 42806	.814
.187	.00766 95373	.00596 27721	.00752 55784	.00448 69221	.01178 03510	.813
.188	.00770 05649	.00599 17624	.00751 48901	.00450 34961	.01179 59498	.812
.189	.00773 14787	.00602 07068	.00750 38509	.00451 99558	.01181 10778	.811
0.190	.00776 22787	.00604 96051	−.00749 24633	.00453 63008	.01182 57359	0.810
.191	.00779 29647	.00607 84570	.00748 07300	.00455 25306	.01183 99249	.809
.192	.00782 35368	.00610 72623	.00746 86536	.00456 86446	.01185 36458	.808
.193	.00785 39947	.00613 60208	.00745 62366	.00458 46424	.01186 68995	.807
.194	.00788 43386	.00616 47323	.00744 34817	.00460 05236	.01187 96867	.806
0.195	.00791 45681	.00619 33965	−.00743 03915	.00461 62876	.01189 20085	0.805
.196	.00794 46834	.00622 20131	.00741 69684	.00463 19339	.01190 38657	.804
.197	.00797 46843	.00625 05820	.00740 32152	.00464 74621	.01191 52591	.803
.198	.00800 45708	.00627 91030	.00738 91344	.00466 28717	.01192 61899	.802
.199	.00803 43427	.00630 75757	.00737 47285	.00467 81623	.01193 66587	.801
0.200	.00806 40000	.00633 60000	−.00736 00000	.00469 33333	.01194 66667	0.800
	$-B(m)$	$-A(m)$	$D(m)$	$-C(m)$	$-E(m)$	m

TABLE I: COEFFICIENTS OF FOURTH ORDER TERMS—*Continued*

$$x = x_0 + ph;\ [F(x) - F(x_0)]/\Delta = m;\ p = m + \tfrac{1}{2}(1-m)[1-(1-m)^2]\frac{\delta_0^2}{\Delta} + \tfrac{1}{2}[m(1-m^2)]\frac{\delta_1^2}{\Delta}$$
$$+ A(m)\frac{\delta_0^4}{\Delta} + B(m)\frac{\delta_1^4}{\Delta} + C(m)\left(\frac{\delta_0^3}{\Delta}\right)^2 + D(m)\left(\frac{\delta_1^3}{\Delta}\right)^2 + E(m)\frac{\delta_0^2}{\Delta}\frac{\delta_1^2}{\Delta} + \text{Sixth Order Terms} + \ldots$$

m	$-A(m)$	$-B(m)$	$-C(m)$	$D(m)$	$E(m)$	
0.200	.00806 40000	.00633 60000	−.00736 00000	.00469 33333	.01194 66667	0.800
.201	.00809 35427	.00636 43756	.00734 49515	.00470 83844	.01195 62146	.799
.202	.00812 29706	.00639 27024	.00732 95856	.00472 33149	.01196 53035	.798
.203	.00815 22837	.00642 09800	.00731 39047	.00473 81246	.01197 39342	.797
.204	.00818 14819	.00644 92082	.00729 79114	.00475 28128	.01198 21078	.796
0.205	.00821 05652	.00647 73869	−.00728 16081	.00476 73792	.01198 98252	0.795
.206	.00823 95335	.00650 55157	.00726 49974	.00478 18233	.01199 70873	.794
.207	.00826 83867	.00653 35945	.00724 80818	.00479 61446	.01200 38951	.793
.208	.00829 71248	.00656 16231	.00723 08637	.00481 03427	.01201 02496	.792
.209	.00832 57477	.00658 96011	.00721 33456	.00482 44172	.01201 61517	.791
0.210	.00835 42553	.00661 75284	−.00719 55300	.00483 83675	.01202 16025	0.790
.211	.00838 26476	.00664 54048	.00717 74193	.00485 21933	.01202 66029	.789
.212	.00841 09245	.00667 32299	.00715 90160	.00486 58940	.01203 11539	.788
.213	.00843 90860	.00670 10037	.00714 03226	.00487 94693	.01203 52566	.787
.214	.00846 71319	.00672 87258	.00712 13414	.00489 29187	.01203 89119	.786
0.215	.00849 50623	.00675 63961	−.00710 20749	.00490 62418	.01204 21209	0.785
.216	.00852 28771	.00678 40142	.00708 25254	.00491 94381	.01204 48845	.784
.217	.00855 05761	.00681 15800	.00706 26955	.00493 25072	.01204 72038	.783
.218	.00857 81594	.00683 90933	.00704 25876	.00494 54486	.01204 90798	.782
.219	.00860 56270	.00686 65538	.00702 22039	.00495 82620	.01205 05136	.781
0.220	.00863 29786	.00689 39614	−.00700 15469	.00497 09469	.01205 15061	0.780
.221	.00866 02144	.00692 13156	.00698 06190	.00498 35029	.01205 20585	.779
.222	.00868 73342	.00694 86165	.00695 94226	.00499 59296	.01205 21718	.778
.223	.00871 43380	.00697 58637	.00693 79600	.00500 82264	.01205 18471	.777
.224	.00874 12257	.00700 30569	.00691 62335	.00502 03931	.01205 10853	.776
0.225	.00876 79973	.00703 01960	−.00689 42456	.00503 24292	.01204 98877	0.775
.226	.00879 46527	.00705 72808	.00687 19985	.00504 43343	.01204 82552	.774
.227	.00882 11920	.00708 43110	.00684 94946	.00505 61079	.01204 61889	.773
.228	.00884 76149	.00711 12864	.00682 67363	.00506 77497	.01204 36900	.772
.229	.00887 39215	.00713 82068	.00680 37257	.00507 92593	.01204 07595	.771
0.230	.00890 01118	.00716 50720	−.00678 04654	.00509 06362	.01203 73985	0.770
.231	.00892 61857	.00719 18816	.00675 69574	.00510 18800	.01203 36080	.769
.232	.00895 21430	.00721 86356	.00673 32042	.00511 29904	.01202 93894	.768
.233	.00897 79839	.00724 53336	.00670 92080	.00512 39670	.01202 47435	.767
.234	.00900 37083	.00727 19755	.00668 49711	.00513 48093	.01201 96716	.766
0.235	.00902 93160	.00729 85611	−.00666 04958	.00514 55169	.01201 41747	0.765
.236	.00905 48072	.00732 50900	.00663 57844	.00515 60895	.01200 82541	.764
.237	.00908 01816	.00735 15622	.00661 08390	.00516 65267	.01200 19108	.763
.238	.00910 54393	.00737 79773	.00658 56620	.00517 68281	.01199 51459	.762
.239	.00913 05803	.00740 43351	.00656 02555	.00518 69933	.01198 79607	.761
0.240	.00915 56045	.00743 06355	−.00653 46219	.00519 70219	.01198 03563	0.760
.241	.00918 05118	.00745 68782	.00650 87633	.00520 69135	.01197 23337	.759
.242	.00920 53023	.00748 30630	.00648 26819	.00521 66678	.01196 38943	.758
.243	.00922 99758	.00750 91896	.00645 63800	.00522 62844	.01195 50391	.757
.244	.00925 45324	.00753 52579	.00642 98598	.00523 57629	.01194 57693	.756
0.245	.00927 89720	.00756 12676	−.00640 31235	.00524 51029	.01193 60861	0.755
.246	.00930 32946	.00758 72185	.00637 61732	.00525 43042	.01192 59906	.754
.247	.00932 75001	.00761 31104	.00634 90111	.00526 33662	.01191 54842	.753
.248	.00935 15886	.00763 89430	.00632 16395	.00527 22887	.01190 45678	.752
.249	.00937 55599	.00766 47162	.00629 40604	.00528 10712	.01189 32429	.751
0.250	.00939 94141	.00769 04297	−.00626 62760	.00528 97135	.01188 15104	0.750
	$-B(m)$	$-A(m)$	$D(m)$	$-C(m)$	$-E(m)$	m

TABLE I: COEFFICIENTS OF FOURTH ORDER TERMS—*Continued*

$$x = x_0 + ph;\ [F(x) - F(x_0)]/\Delta = m;\ p = m + \tfrac{1}{6}(1-m)[1-(1-m)^2]\frac{\delta_0^2}{\Delta} + \tfrac{1}{6}[m(1-m^2)]\frac{\delta_1^2}{\Delta}$$

$$+ A(m)\frac{\delta_0^4}{\Delta} + B(m)\frac{\delta_1^4}{\Delta} + C(m)\left(\frac{\delta_0^2}{\Delta}\right)^2 + D(m)\left(\frac{\delta_1^2}{\Delta}\right)^2 + E(m)\frac{\delta_0^2}{\Delta}\frac{\delta_1^2}{\Delta} + \text{Sixth Order Terms} + \ldots$$

m	$-A(m)$	$-B(m)$	$-C(m)$	$D(m)$	$E(m)$	
0.250	.00939 94141	.00769 04297	−.00626 62760	.00528 97135	.01188 15104	0.750
.251	.00942 31510	.00771 60833	.00623 82885	.00529 82152	.01186 93717	.749
.252	.00944 67708	.00774 16768	.00621 01001	.00530 65759	.01185 68280	.748
.253	.00947 02733	.00776 72100	.00618 17128	.00531 47953	.01184 38805	.747
.254	.00949 36585	.00779 26826	.00615 31287	.00532 28730	.01183 05303	.746
0.255	.00951 69265	.00781 80944	−.00612 43501	.00533 08087	.01181 67788	0.745
.256	.00954 00770	.00784 34453	.00609 53789	.00533 86020	.01180 26271	.744
.257	.00956 31103	.00786 87349	.00606 62174	.00534 62526	.01178 80765	.743
.258	.00958 60261	.00789 39631	.00603 68676	.00535 37601	.01177 31283	.742
.259	.00960 88245	.00791 91297	.00600 73316	.00536 11243	.01175 77836	.741
0.260	.00963 15055	.00794 42345	−.00597 76115	.00536 83448	.01174 20437	0.740
.261	.00965 40690	.00796 92771	.00594 77093	.00537 54212	.01172 59100	.739
.262	.00967 65151	.00799 42575	.00591 76272	.00538 23533	.01170 93835	.738
.263	.00969 88436	.00801 91754	.00588 73671	.00538 91407	.01169 24657	.737
.264	.00972 10546	.00804 40306	.00585 69312	.00539 57830	.01167 51577	.736
0.265	.00974 31481	.00806 88228	−.00582 63215	.00540 22801	.01165 74609	0.735
.266	.00976 51239	.00809 35519	.00579 55400	.00540 86315	.01163 93766	.734
.267	.00978 69822	.00811 82176	.00576 45887	.00541 48369	.01162 09059	.733
.268	.00980 87229	.00814 28197	.00573 34698	.00542 08960	.01160 20502	.732
.269	.00983 03459	.00816 73581	.00570 21851	.00542 68086	.01158 28108	.731
0.270	.00985 18513	.00819 18324	−.00567 07367	.00543 25742	.01156 31890	0.730
.271	.00987 32391	.00821 62425	.00563 91267	.00543 81927	.01154 31862	.729
.272	.00989 45091	.00824 05882	.00560 73569	.00544 36637	.01152 28035	.728
.273	.00991 56615	.00826 48693	.00557 54294	.00544 89868	.01150 20422	.727
.274	.00993 66961	.00828 90855	.00554 33462	.00545 41619	.01148 09039	.726
0.275	.00995 76130	.00831 32366	−.00551 11092	.00545 91886	.01145 93896	0.725
.276	.00997 84122	.00833 73224	.00547 87204	.00546 40667	.01143 75009	.724
.277	.00999 90937	.00836 13427	.00544 61818	.00546 87958	.01141 52389	.723
.278	.01001 96573	.00838 52974	.00541 34953	.00547 33756	.01139 26051	.722
.279	.01004 01032	.00840 91861	.00538 06629	.00547 78059	.01136 96007	.721
0.280	.01006 04314	.00843 30086	−.00534 76864	.00548 20864	.01134 62272	0.720
.281	.01008 06417	.00845 67649	.00531 45679	.00548 62168	.01132 24858	.719
.282	.01010 07342	.00848 04546	.00528 13092	.00549 01969	.01129 83779	.718
.283	.01012 07089	.00850 40775	.00524 79122	.00549 40263	.01127 39049	.717
.284	.01014 05658	.00852 76334	.00521 43789	.00549 77049	.01124 90682	.716
0.285	.01016 03049	.00855 11222	−.00518 07112	.00550 12323	.01122 38690	0.715
.286	.01017 99262	.00857 45436	.00514 69109	.00550 46083	.01119 83088	.714
.287	.01019 94296	.00859 78973	.00511 29800	.00550 78326	.01117 23889	.713
.288	.01021 88152	.00862 11833	.00507 89203	.00551 09050	.01114 61107	.712
.289	.01023 80829	.00864 44012	.00504 47337	.00551 38252	.01111 94756	.711
0.290	.01025 72328	.00866 75510	−.00501 04221	.00551 65929	.01109 24850	0.710
.291	.01027 62648	.00869 06322	.00497 59872	.00551 92080	.01106 51403	.709
.292	.01029 51790	.00871 36449	.00494 14311	.00552 16701	.01103 74428	.708
.293	.01031 39754	.00873 65887	.00490 67554	.00552 39791	.01100 93940	.707
.294	.01033 26539	.00875 94634	.00487 19621	.00552 61346	.01098 09953	.706
0.295	.01035 12145	.00878 22689	−.00483 70529	.00552 81365	.01095 22480	0.705
.296	.01036 96573	.00880 50049	.00480 20298	.00552 99846	.01092 31536	.704
.297	.01038 79822	.00882 76712	.00476 68944	.00553 16785	.01089 37136	.703
.298	.01040 61893	.00885 02676	.00473 16487	.00553 32181	.01086 39292	.702
.299	.01042 42786	.00887 27940	.00469 62944	.00553 46031	.01083 38020	.701
0.300	.01044 22500	.00889 52500	−.00466 08333	.00553 58333	.01080 33333	0.700
	$-B(m)$	$-A(m)$	$D(m)$	$-C(m)$	$-E(m)$	m

TABLE I: COEFFICIENTS OF FOURTH ORDER TERMS—*Continued*

$$x = x_0 + ph;\ [F(x) - F(x_0)]/\Delta = m;\ p = m + \tfrac{1}{2}(1-m)[1 - (1-m)^2]\frac{\delta_0^2}{\Delta} + \tfrac{1}{3}[m(1-m^2)]\frac{\delta_0^2}{\Delta}$$

$$+ A(m)\frac{\delta_0^4}{\Delta} + B(m)\frac{\delta_1^4}{\Delta^1} + C(m)\left(\frac{\delta_0^2}{\Delta}\right)^2 + D(m)\left(\frac{\delta_1^2}{\Delta}\right)^2 + E(m)\frac{\delta_0^2}{\Delta}\frac{\delta_1^2}{\Delta} + \text{Sixth Order Terms} + \ldots$$

m	$-A(m)$	$-B(m)$	$-C(m)$	$D(m)$	$E(m)$	
0.300	.01044 22500	.00889 52500	−.00466 08333	.00553 58333	.01080 33333	0.700
.301	.01046 01036	.00891 76355	.00462 52672	.00553 69086	.01077 25247	.699
.302	.01047 78393	.00893 99504	.00458 95979	.00553 78286	.01074 13775	.698
.303	.01049 54573	.00896 21943	.00455 38272	.00553 85932	.01070 98931	.697
.304	.01051 29574	.00898 43671	.00451 79567	.00553 92022	.01067 80731	.696
0.305	.01053 03397	.00900 64687	−.00448 19884	.00553 96553	.01064 59188	0.695
.306	.01054 76042	.00902 84986	.00444 59238	.00553 99524	.01061 34318	.694
.307	.01056 47509	.00905 04569	.00440 97649	.00554 00932	.01058 06134	.693
.308	.01058 17799	.00907 23432	.00437 35132	.00554 00776	.01054 74652	.692
.309	.01059 86910	.00909 41574	.00433 71706	.00553 99053	.01051 39886	.691
0.310	.01061 54845	.00911 58993	−.00430 07388	.00553 95763	.01048 01850	0.690
.311	.01063 21601	.00913 75686	.00426 42194	.00553 90902	.01044 60559	.689
.312	.01064 87180	.00915 91652	.00422 76142	.00553 84469	.01041 16029	.688
.313	.01066 51582	.00918 06889	.00419 09250	.00553 76462	.01037 68273	.687
.314	.01068 14807	.00920 21394	.00415 41533	.00553 66880	.01034 17307	.686
0.315	.01069 76855	.00922 35166	−.00411 73009	.00553 55720	.01030 63146	0.685
.316	.01071 37726	.00924 48202	.00408 03695	.00553 42981	.01027 05804	.684
.317	.01072 97421	.00926 60501	.00404 33607	.00553 28662	.01023 45296	.683
.318	.01074 55939	.00928 72061	.00400 62763	.00553 12760	.01019 81637	.682
.319	.01076 13281	.00930 82879	.00396 91178	.00552 95274	.01016 14843	.681
0.320	.01077 69446	.00932 92954	−.00393 18869	.00552 76203	.01012 44928	0.680
.321	.01079 24436	.00935 02283	.00389 45854	.00552 55544	.01008 71908	.679
.322	.01080 78250	.00937 10865	.00385 72147	.00552 33296	.01004 95797	.678
.323	.01082 30888	.00939 18698	.00381 97766	.00552 09459	.01001 16610	.677
.324	.01083 82351	.00941 25779	.00378 22727	.00551 84029	.00997 34364	.676
0.325	.01085 32639	.00943 32107	−.00374 47046	.00551 57007	.00993 49072	0.675
.326	.01086 81752	.00945 37680	.00370 70739	.00551 28390	.00989 60751	.674
.327	.01088 29690	.00947 42495	.00366 93822	.00550 98177	.00985 69416	.673
.328	.01089 76454	.00949 46551	.00363 16312	.00550 66367	.00981 75081	.672
.329	.01091 22044	.00951 49846	.00359 38224	.00550 32958	.00977 77763	.671
0.330	.01092 66459	.00953 52378	−.00355 59574	.00549 97949	.00973 77476	0.670
.331	.01094 09701	.00955 54145	.00351 80378	.00549 61340	.00969 74237	.669
.332	.01095 51769	.00957 55145	.00348 00652	.00549 23128	.00965 68060	.668
.333	.01096 92665	.00959 55376	.00344 20411	.00548 83313	.00961 58961	.667
.334	.01098 32387	.00961 54835	.00340 39672	.00548 41893	.00957 46956	.666
0.335	.01099 70936	.00963 53522	−.00336 58448	.00547 98867	.00953 32059	0.665
.336	.01101 08314	.00965 51434	.00332 76757	.00547 54235	.00949 14288	.664
.337	.01102 44519	.00967 48569	.00328 94613	.00547 07995	.00944 93657	.663
.338	.01103 79552	.00969 44926	.00325 12032	.00546 60146	.00940 70182	.662
.339	.01105 13414	.00971 40502	.00321 29029	.00546 10688	.00936 43879	.661
0.340	.01106 46105	.00973 35295	−.00317 45619	.00545 59619	.00932 14763	0.660
.341	.01107 77625	.00975 29304	.00313 61817	.00545 06938	.00927 82850	.659
.342	.01109 07974	.00977 22526	.00309 77639	.00544 52645	.00923 48156	.658
.343	.01110 37153	.00979 14961	.00305 93100	.00543 96738	.00919 10698	.657
.344	.01111 65163	.00981 06604	.00302 08213	.00543 39217	.00914 70489	.656
0.345	.01112 92003	.00982 97456	−.00298 22996	.00542 80082	.00910 27548	0.655
.346	.01114 17673	.00984 87514	.00294 37461	.00542 19330	.00905 81888	.654
.347	.01115 42175	.00986 76775	.00290 51624	.00541 56963	.00901 33528	.653
.348	.01116 65509	.00988 65239	.00286 65500	.00540 92979	.00896 82481	.652
.349	.01117 87674	.00990 52903	.00282 79103	.00540 27376	.00892 28765	.651
0.350	.01119 08672	.00992 39766	−.00278 92448	.00539 60156	.00887 72396	0.650
	$-B(m)$	$-A(m)$	$D(m)$	$-C(m)$	$-E(m)$	m

TABLE I: COEFFICIENTS OF FOURTH ORDER TERMS—*Continued*

$$x = x_0 + ph;\ [F(x) - F(x_0)]/\Delta = m;\ p = m + \tfrac{1}{2}(1-m)[1-(1-m)^2]\frac{\delta_0^2}{\Delta} + \tfrac{1}{2}[m(1-m^2)]\frac{\delta_1^2}{\Delta}$$
$$+ A(m)\frac{\delta_0^4}{\Delta} + B(m)\frac{\delta_1^4}{\Delta} + C(m)\left(\frac{\delta_0^2}{\Delta}\right)^2 + D(m)\left(\frac{\delta_1^2}{\Delta}\right)^2 + E(m)\frac{\delta_0^2}{\Delta}\frac{\delta_1^2}{\Delta} + \text{Sixth Order Terms} + \ldots$$

m	$-A(m)$	$-B(m)$	$-C(m)$	$D(m)$	$E(m)$	
0.350	.01119 08672	.00992 39766	−.00278 92448	.00539 60156	.00887 72396	0.650
.351	.01120 28502	.00994 25824	.00275 05549	.00538 91317	.00883 13389	.649
.352	.01121 47166	.00996 11078	.00271 18421	.00538 20859	.00878 51760	.648
.353	.01122 64663	.00997 95524	.00267 31077	.00537 48781	.00873 87527	.647
.354	.01123 80994	.00999 79161	.00263 43533	.00536 75083	.00869 20704	.646
0.355	.01124 96159	.01001 61987	−.00259 55803	.00535 99764	.00864 51308	0.645
.356	.01126 10159	.01003 44000	.00255 67900	.00535 22824	.00859 79356	.644
.357	.01127 22994	.01005 25198	.00251 79839	.00534 44263	.00855 04863	.643
.358	.01128 34665	.01007 05579	.00247 91634	.00533 64080	.00850 27846	.642
.359	.01129 45172	.01008 85142	.00244 03299	.00532 82275	.00845 48320	.641
0.360	.01130 54515	.01010 63885	−.00240 14848	.00531 98848	.00840 66304	0.640
.361	.01131 62695	.01012 41805	.00236 26294	.00531 13799	.00835 81812	.639
.362	.01132 69713	.01014 18901	.00232 37652	.00530 27127	.00830 94862	.638
.363	.01133 75568	.01015 95172	.00228 48934	.00529 38832	.00826 05469	.637
.364	.01134 80262	.01017 70614	.00224 60156	.00528 48914	.00821 13650	.636
0.365	.01135 83794	.01019 45227	−.00220 71330	.00527 57374	.00816 19421	0.635
.366	.01136 86166	.01021 19008	.00216 82469	.00526 64211	.00811 22800	.634
.367	.01137 87377	.01022 91957	.00212 93588	.00525 69425	.00806 23802	.633
.368	.01138 87429	.01024 64070	.00209 04700	.00524 73016	.00801 22444	.632
.369	.01139 86321	.01026 35346	.00205 15818	.00523 74984	.00796 18743	.631
0.370	.01140 84055	.01028 05783	−.00201 26955	.00522 75330	.00791 12716	0.630
.371	.01141 80630	.01029 75380	.00197 38124	.00521 74053	.00786 04378	.629
.372	.01142 76047	.01031 44134	.00193 49340	.00520 71154	.00780 93746	.628
.373	.01143 70308	.01033 12044	.00189 60614	.00519 66633	.00775 80838	.627
.374	.01144 63411	.01034 79108	.00185 71960	.00518 60490	.00770 65670	.626
0.375	.01145 55359	.01036 45325	−.00181 83390	.00517 52726	.00765 48259	0.625
.376	.01146 46151	.01038 10691	.00177 94919	.00516 43341	.00760 28621	.624
.377	.01147 35788	.01039 75207	.00174 06557	.00515 32335	.00755 06773	.623
.378	.01148 24270	.01041 38869	.00170 18319	.00514 19709	.00749 82732	.622
.379	.01149 11599	.01043 01676	.00166 30217	.00513 05463	.00744 56515	.621
0.380	.01149 97774	.01044 63626	−.00162 42264	.00511 89597	.00739 28139	0.620
.381	.01150 82796	.01046 24718	.00158 54472	.00510 72113	.00733 97620	.619
.382	.01151 66666	.01047 84950	.00154 66853	.00509 53011	.00728 64976	.618
.383	.01152 49384	.01049 44319	.00150 79421	.00508 32291	.00723 30223	.617
.384	.01153 30952	.01051 02825	.00146 92187	.00507 09954	.00717 93379	.616
0.385	.01154 11369	.01052 60465	−.00143 05164	.00505 86000	.00712 54460	0.615
.386	.01154 90636	.01054 17237	.00139 18365	.00504 60432	.00707 13483	.614
.387	.01155 68754	.01055 73141	.00135 31801	.00503 33248	.00701 70466	.613
.388	.01156 45724	.01057 28173	.00131 45484	.00502 04450	.00696 25426	.612
.389	.01157 21545	.01058 82333	.00127 59427	.00500 74040	.00690 78379	.611
0.390	.01157 96219	.01060 35618	−.00123 73642	.00499 42017	.00685 29342	0.610
.391	.01158 69747	.01061 88028	.00119 88140	.00498 08382	.00679 78333	.609
.392	.01159 42128	.01063 39559	.00116 02934	.00496 73137	.00674 25369	.608
.393	.01160 13364	.01064 90211	.00112 18035	.00495 36283	.00668 70467	.607
.394	.01160 83455	.01066 39981	.00108 33455	.00493 97821	.00663 13644	.606
0.395	.01161 52402	.01067 88869	−.00104 49207	.00492 57751	.00657 54917	0.605
.396	.01162 20206	.01069 36872	.00100 65301	.00491 16075	.00651 94304	.604
.397	.01162 86867	.01070 83988	.00096 81749	.00489 72794	.00646 31822	.603
.398	.01163 52386	.01072 30216	.00092 98563	.00488 27909	.00640 67488	.602
.399	.01164 16763	.01073 75554	.00089 15754	.00486 81422	.00635 01319	.601
0.400	.01164 80000	.01075 20000	−.00085 33333	.00485 33333	.00629 33333	0.600
	$-B(m)$	$-A(m)$	$D(m)$	$-C(m)$	$-E(m)$	m

TABLE I: COEFFICIENTS OF FOURTH ORDER TERMS—*Continued*

$$x = x_0 + ph;\ [F(x) - F(x_0)]/\Delta = m;\ p = m + \tfrac{1}{2}(1-m)[1-(1-m)^2]\frac{\delta_0^2}{\Delta} + \tfrac{1}{3}[m(1-m^3)]\frac{\delta_1^2}{\Delta}$$

$$+ A(m)\frac{\delta_0^4}{\Delta} + B(m)\frac{\delta_1^4}{\Delta} + C(m)\left(\frac{\delta_0^2}{\Delta}\right)^2 + D(m)\left(\frac{\delta_1^2}{\Delta}\right)^2 + E(m)\frac{\delta_0^2}{\Delta}\frac{\delta_1^2}{\Delta} + \text{Sixth Order Terms} + \ldots$$

m	$-A(m)$	$-B(m)$	$-C(m)$	$D(m)$	$E(m)$	
0.400	.01164 80000	.01075 20000	−.00085 33333	.00485 33333	.00629 33333	0.600
.401	.01165 42097	.01076 63553	.00081 51313	.00483 83645	.00623 63547	.599
.402	.01166 03054	.01078 06211	.00077 69704	.00482 32357	.00617 91979	.598
.403	.01166 62873	.01079 47972	.00073 88517	.00480 79473	.00612 18645	.597
.404	.01167 21554	.01080 88835	.00070 07765	.00479 24992	.00606 43563	.596
0.405	.01167 79098	.01082 28798	−.00066 27457	.00477 68918	.00600 66750	0.595
.406	.01168 35506	.01083 67859	.00062 47605	.00476 11250	.00594 88225	.594
.407	.01168 90777	.01085 06017	.00058 68220	.00474 51991	.00589 08004	.593
.408	.01169 44914	.01086 43269	.00054 89312	.00472 91143	.00583 26105	.592
.409	.01169 97917	.01087 79615	.00051 10894	.00471 28706	.00577 42545	.591
0.410	.01170 49786	.01089 15052	−.00047 32975	.00469 64683	.00571 57342	0.590
.411	.01171 00522	.01090 49579	.00043 55567	.00467 99076	.00565 70513	.589
.412	.01171 50127	.01091 83194	.00039 78679	.00466 31885	.00559 82076	.588
.413	.01171 98600	.01093 15896	.00036 02323	.00464 63114	.00553 92048	.587
.414	.01172 45943	.01094 47682	.00032 26510	.00462 92763	.00548 00447	.586
0.415	.01172 92156	.01095 78552	−.00028 51249	.00461 20835	.00542 07291	0.585
.416	.01173 37241	.01097 08504	.00024 76552	.00459 47331	.00536 12597	.584
.417	.01173 81197	.01098 37535	.00021 02428	.00457 72254	.00530 16382	.583
.418	.01174 24026	.01099 65645	.00017 28888	.00455 95606	.00524 18666	.582
.419	.01174 65729	.01100 92832	.00013 55943	.00454 17388	.00518 19464	.581
0.420	.01175 06306	.01102 19094	−.00009 83603	.00452 37603	.00512 18795	0.580
.421	.01175 45759	.01103 44429	−.00006 11877	.00450 56252	.00506 16676	.579
.422	.01175 84087	.01104 68836	−.00002 40776	.00448 73339	.00500 13125	.578
.423	.01176 21292	.01105 92313	+.00001 29690	.00446 88864	.00494 08160	.577
.424	.01176 57375	.01107 14859	.00004 99511	.00445 02832	.00488 01799	.576
0.425	.01176 92336	.01108 36472	.00008 68677	.00443 15243	.00481 94059	0.575
.426	.01177 26177	.01109 57151	.00012 37178	.00441 26100	.00475 84958	.574
.427	.01177 58898	.01110 76893	.00016 05004	.00439 35405	.00469 74514	.573
.428	.01177 90500	.01111 95697	.00019 72146	.00437 43161	.00463 62745	.572
.429	.01178 20984	.01113 13563	.00023 38594	.00435 49371	.00457 49668	.571
0.430	.01178 50350	.01114 30487	.00027 04338	.00433 54037	.00451 35301	0.570
.431	.01178 78601	.01115 46469	.00030 69369	.00431 57161	.00445 19662	.569
.432	.01179 05736	.01116 61507	.00034 33676	.00429 58746	.00439 02770	.568
.433	.01179 31756	.01117 75599	.00037 97251	.00427 58795	.00432 84641	.567
.434	.01179 56662	.01118 88744	.00041 60084	.00425 57310	.00426 65294	.566
0.435	.01179 80456	.01120 00940	.00045 22166	.00423 54295	.00420 44746	0.565
.436	.01180 03138	.01121 12186	.00048 83487	.00421 49751	.00414 23016	.564
.437	.01180 24709	.01122 22480	.00052 44039	.00419 43682	.00408 00122	.563
.438	.01180 45170	.01123 31820	.00056 03811	.00417 36091	.00401 76080	.562
.439	.01180 64521	.01124 40206	.00059 62795	.00415 26980	.00395 50910	.561
0.440	.01180 82765	.01125 47635	.00063 20981	.00413 16352	.00389 24629	0.560
.441	.01180 99901	.01126 54106	.00066 78362	.00411 04211	.00382 97256	.559
.442	.01181 15931	.01127 59618	.00070 34926	.00408 90559	.00376 68807	.558
.443	.01181 30855	.01128 64168	.00073 90667	.00406 75400	.00370 39302	.557
.444	.01181 44675	.01129 67756	.00077 45574	.00404 58736	.00364 08758	.556
0.445	.01181 57391	.01130 70380	.00080 99639	.00402 40572	.00357 77193	0.555
.446	.01181 69005	.01131 72038	.00084 52854	.00400 20909	.00351 44625	.554
.447	.01181 79517	.01132 72729	.00088 05209	.00397 99751	.00345 11073	.553
.448	.01181 88928	.01133 72452	.00091 56696	.00395 77102	.00338 76554	.552
.449	.01181 97240	.01134 71204	.00095 07306	.00393 52965	.00332 41086	.551
0.450	.01182 04453	.01135 68984	.00098 57031	.00391 27344	.00326 04687	0.550
	$-B(m)$	$-A(m)$	$D(m)$	$-C(m)$	$-E(m)$	m

TABLE I: COEFFICIENTS OF FOURTH ORDER TERMS—*Continued*

$$x = x_0 + ph; [F(x) - F(x_0)]/\Delta = m; p = m + \tfrac{1}{2}(1-m)[1-(1-m)^2]\frac{\delta_0^2}{\Delta} + \tfrac{1}{2}[m(1-m^2)]\frac{\delta_1^2}{\Delta}$$
$$+ A(m)\frac{\delta_0^4}{\Delta} + B(m)\frac{\delta_1^4}{\Delta} + C(m)\left(\frac{\delta_0^2}{\Delta}\right)^2 + D(m)\left(\frac{\delta_1^2}{\Delta}\right)^2 + E(m)\frac{\delta_0^2}{\Delta}\frac{\delta_1^2}{\Delta} + \text{Sixth Order Terms} + \ldots$$

m	$-A(m)$	$-B(m)$	$-C(m)$	$D(m)$	$E(m)$	
0.450	.01182 04453	.01135 68984	.00098 57031	.00391 27344	.00326 04687	0.550
.451	.01182 10568	.01136 65792	.00102 05863	.00389 00241	.00319 67377	.549
.452	.01182 15587	.01137 61625	.00105 53792	.00386 71660	.00313 29171	.548
.453	.01182 19510	.01138 56481	.00109 00810	.00384 41605	.00306 90090	.547
.454	.01182 22338	.01139 50361	.00112 46910	.00382 10080	.00300 50150	.546
0.455	.01182 24073	.01140 43261	.00115 92082	.00379 77087	.00294 09370	0.545
.456	.01182 24715	.01141 35180	.00119 36319	.00377 42631	.00287 67768	.544
.457	.01182 24265	.01142 26118	.00122 79613	.00375 06715	.00281 25363	.543
.458	.01182 22724	.01143 16072	.00126 21955	.00372 69344	.00274 82171	.542
.459	.01182 20094	.01144 05042	.00129 63337	.00370 30520	.00268 38212	.541
0.460	.01182 16375	.01144 93025	.00133 03752	.00367 90248	.00261 93504	0.540
.461	.01182 11569	.01145 80020	.00136 43191	.00365 48532	.00255 48065	.539
.462	.01182 05676	.01146 66026	.00139 81647	.00363 05375	.00249 01912	.538
.463	.01181 98698	.01147 51042	.00143 19111	.00360 60781	.00242 55065	.537
.464	.01181 90635	.01148 35065	.00146 55576	.00358 14755	.00236 07541	.536
0.465	.01181 81489	.01149 18095	.00149 91035	.00355 67301	.00229 59358	0.535
.466	.01181 71261	.01150 00130	.00153 25479	.00353 18423	.00223 10536	.534
.467	.01181 59951	.01150 81168	.00156 58900	.00350 68124	.00216 61091	.533
.468	.01181 47561	.01151 61209	.00159 91293	.00348 16410	.00210 11043	.532
.469	.01181 34092	.01152 40251	.00163 22647	.00345 63284	.00203 60409	.531
0.470	.01181 19546	.01153 18292	.00166 52958	.00343 08751	.00197 09207	0.530
.471	.01181 03922	.01153 95331	.00169 82216	.00340 52814	.00190 57457	.529
.472	.01180 87223	.01154 71367	.00173 10415	.00337 95480	.00184 05175	.528
.473	.01180 69449	.01155 46398	.00176 37547	.00335 36751	.00177 52381	.527
.474	.01180 50601	.01156 20423	.00179 63605	.00332 76632	.00170 99093	.526
0.475	.01180 30681	.01156 93440	.00182 88582	.00330 15129	.00164 45329	0.525
.476	.01180 09690	.01157 65448	.00186 12471	.00327 52245	.00157 91107	.524
.477	.01179 87628	.01158 36447	.00189 35265	.00324 87985	.00151 36445	.523
.478	.01179 64498	.01159 06433	.00192 56956	.00322 22354	.00144 81362	.522
.479	.01179 40299	.01159 75407	.00195 77537	.00319 55356	.00138 25876	.521
0.480	.01179 15034	.01160 43366	.00198 97003	.00316 86997	.00131 70005	0.520
.481	.01178 88703	.01161 10310	.00202 15345	.00314 17282	.00125 13768	.519
.482	.01178 61307	.01161 76237	.00205 32557	.00311 46214	.00118 57183	.518
.483	.01178 32848	.01162 41145	.00208 48632	.00308 73799	.00112 00268	.517
.484	.01178 03327	.01163 05034	.00211 63564	.00306 00042	.00105 43042	.516
0.485	.01177 72745	.01163 67901	.00214 77346	.00303 24949	.00098 85522	0.515
.486	.01177 41103	.01164 29746	.00217 89971	.00300 48523	.00092 27728	.514
.487	.01177 08402	.01164 90568	.00221 01432	.00297 70770	.00085 69676	.513
.488	.01176 74644	.01165 50364	.00224 11723	.00294 91696	.00079 11387	.512
.489	.01176 39830	.01166 09134	.00227 20839	.00292 11306	.00072 52878	.511
0.490	.01176 03960	.01166 66877	.00230 28771	.00289 29604	.00065 94167	0.510
.491	.01175 67037	.01167 23591	.00233 35514	.00286 46597	.00059 35273	.509
.492	.01175 29061	.01167 79274	.00236 41062	.00283 62289	.00052 76213	.508
.493	.01174 90033	.01168 33926	.00239 45408	.00280 76685	.00046 17008	.507
.494	.01174 49955	.01168 87545	.00242 48546	.00277 89793	.00039 57673	.506
0.495	.01174 08828	.01169 40130	.00245 50470	.00275 01616	.00032 98229	0.505
.496	.01173 66653	.01169 91680	.00248 51174	.00272 12161	.00026 38693	.504
.497	.01173 23432	.01170 42193	.00251 50651	.00269 21432	.00019 79084	.503
.498	.01172 79165	.01170 91668	.00254 48897	.00266 29437	.00013 19420	.502
.499	.01172 33854	.01171 40104	.00257 45904	.00263 36180	.00006 59719	.501
0.500	.01171 87500	.01171 87500	.00260 41667	.00260 41667	.00000 00000	0.500
	$-B(m)$	$-A(m)$	$D(m)$	$-C(m)$	$-E(m)$	m

TABLE II: COEFFICIENTS OF SIXTH ORDER TERMS

$$p = m + \text{2nd order terms} + \text{4th order terms} + F(m)\frac{\delta_0^6}{\Delta} + G(m)\frac{\delta_1^6}{\Delta} + H(m)\frac{\delta_0^2}{\Delta}\frac{\delta_0^4}{\Delta} + I(m)\frac{\delta_1^2}{\Delta}\frac{\delta_1^4}{\Delta} + J(m)\frac{\delta_0^2}{\Delta}\frac{\delta_1^4}{\Delta} + K(m)\frac{\delta_1^2}{\Delta}\frac{\delta_0^4}{\Delta} + L(m)\left(\frac{\delta_0^2}{\Delta}\right)^3\frac{\delta_1^2}{\Delta} + M(m)\left(\frac{\delta_1^2}{\Delta}\right)^2\frac{\delta_0^2}{\Delta} + N(m)\left(\frac{\delta_0^2}{\Delta}\right)^3 + O(m)\left(\frac{\delta_1^2}{\Delta}\right)^3 + \cdots$$

m	$F(m)$	$G(m)$	$H(m)$	$I(m)$	$J(m)$	m
.0	.00000 000000	.00000 000000	.00000 000	.00000 0000	.00000 000000	.0
.1	.00088 642125	.00070 459125	−.00223 991	−.00106 161	−.00169 90275	.1
.2	.00160 512	.00135 168	−.00265 728	−.00183 808	−.00233 472	.2
.3	.00211 579875	.00188 706375	−.00188 377	−.00210 240$\dot{3}$	−.00203 083416$\dot{6}$	.3
.4	.00239 616	.00226 304	−.00055 637$\dot{3}$	−.00173 824	−.00106 496	.4
.5	.00244 140625	.00244 140625	.00078 125	−.00078 125	.00019 53125	.5
.6	.00226 304	.00239 616	.00173 824	.00055 637$\dot{3}$	.00136 042666$\dot{6}$	.6
.7	.00188 706375	.00211 579875	.00210 240$\dot{3}$	.00188 377	.00208 54925	.7
.8	.00135 168	.00160 512	.00183 808	.00265 728	.00212 992	.8
.9	.00070 459125	.00088 642125	.00106 161	.00223 991	.00140 44525	.9
1.0	.00000 000000	.00000 000000	.00000 000	.00000 0000	.00000 00000	1.0

m	$K(m)$	$L(m)$	$M(m)$	$N(m)$	$O(m)$	m
.0	.00000 00000	.00000 00000	.00000 00000	.00000 00000	.00000 00000	.0
.1	−.00140 44525	.00266 965	.00184 685	.00125 33666$\dot{6}$	.00041 76333$\dot{3}$	.1
.2	−.00212 992	.00145 20888$\dot{8}$	.00175 50222$\dot{2}$	.00020 69333$\dot{3}$	.00058 59555$\dot{5}$	.2
.3	−.00208 54925	−.00101 27833$\dot{3}$	.00021 12833$\dot{3}$	−.00087 39888$\dot{8}$	.00036 29888$\dot{8}$	.3
.4	−.00136 042666$\dot{6}$	−.00281 17333$\dot{3}$	−.00176 42666$\dot{6}$	−.00121 74222$\dot{2}$	−.00020 65777$\dot{7}$	.4
.5	−.00019 53125	−.00303 81944$\dot{4}$	−.00303 81944$\dot{4}$	−.00086 80555$\dot{5}$	−.00086 80555$\dot{5}$	.5
.6	.00106 496	−.00176 42666$\dot{6}$	−.00281 17333$\dot{3}$	−.00020 65777$\dot{7}$	−.00121 74222$\dot{2}$	.6
.7	.00203 083416$\dot{6}$	.00021 12833$\dot{3}$	−.00101 27833$\dot{3}$	.00036 29888$\dot{8}$	−.00087 39888$\dot{8}$	.7
.8	.00233 472	.00175 50222$\dot{2}$	.00145 20888$\dot{8}$	.00058 59555$\dot{5}$	.00020 69333$\dot{3}$	.8
.9	.00169 90275	.00184 685	.00266 965	.00041 76333$\dot{3}$	.00125 33666$\dot{6}$	.9
1.0	.00000 00000	.00000 00000	.00000 00000	.00000 00000	.00000 00000	1.0